AP Calculus AB
Review Materials

Name: ________________

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AP CALCULUS
Stuff you MUST know Cold

* means topic only on BC

Curve sketching and analysis

$y = f(x)$ must be continuous at each:

critical point: $\frac{dy}{dx} = 0$ or undefined and look out for endpoints

local minimum:

$\frac{dy}{dx}$ goes (–,0,+) or (–,und,+) or $\frac{d^2y}{dx^2} > 0$

local maximum:

$\frac{dy}{dx}$ goes (+,0,–) or (+,und,–) or $\frac{d^2y}{dx^2} < 0$

point of inflection: concavity changes

$\frac{d^2y}{dx^2}$ goes from (+,0,–), (–,0,+), (+,und,–), or (–,und,+)

Basic Derivatives

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$
$$\frac{d}{dx}\left(e^u\right) = e^u \frac{du}{dx}$$

where u is a function of x, and a is a constant.

More Derivatives

$$\frac{d}{dx}\left(\sin^{-1}\frac{u}{a}\right) = \frac{1}{\sqrt{a^2-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\tan^{-1}\frac{u}{a}\right) = \frac{a}{a^2+u^2}\cdot\frac{du}{dx}$$
$$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}\left(\sec^{-1}\frac{u}{a}\right) = \frac{a}{|u|\sqrt{u^2-a^2}}\cdot\frac{du}{dx}$$
$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left(a^{u(x)}\right) = a^{u(x)}\ln a\cdot\frac{du}{dx}$$
$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$$

Differentiation Rules

Chain Rule

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx} \quad OR \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Product Rule

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx} \quad OR \quad u'v + uv'$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} \quad OR \quad \frac{u'v-uv'}{v^2}$$

"PLUS A CONSTANT"

The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$

Corollary to FTC

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))b'(x) - f(a(x))a'(x)$$

Intermediate Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, and y is a number between $f(a)$ and $f(b)$, then there exists at least one number $x = c$ in the open interval (a, b) such that

$$f(c) = y.$$

Mean Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, AND the first derivative exists on the interval (a, b), then there is at least one number $x = c$ in (a, b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Rolle's Theorem

If the function $f(x)$ is continuous on $[a, b]$, AND the first derivative exists on the interval (a, b), AND $f(a) = f(b)$, then there is at least one number $x = c$ in (a, b) such that

$$f'(c) = 0.$$

Approx. Methods for Integration

Trapezoidal Rule

$$\int_a^b f(x)dx = \tfrac{1}{2}\tfrac{b-a}{n}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

Simpson's Rule

$$\int_a^b f(x)dx = \tfrac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Theorem of the Mean Value i.e. AVERAGE VALUE

If the function $f(x)$ is continuous on $[a, b]$ and the first derivative exists on the interval (a, b), then there exists a number $x = c$ on (a, b) such that

$$f(c) = \frac{\int_a^b f(x)dx}{(b-a)}$$

This value $f(c)$ is the "average value" of the function on the interval $[a, b]$.

Solids of Revolution and friends

Disk Method

$$V = \pi\int_{x=a}^{x=b}[R(x)]^2\,dx$$

Washer Method

$$V = \pi\int_a^b\left([R(x)]^2 - [r(x)]^2\right)dx$$

General volume equation (not rotated)

$$V = \int_a^b Area(x)\,dx$$

*Arc Length $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$

$$= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

Distance, Velocity, and Acceleration

velocity = $\frac{d}{dt}$(position)

acceleration = $\frac{d}{dt}$(velocity)

*velocity vector = $\left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$

speed = $|v| = \sqrt{(x')^2 + (y')^2}$ *

displacement = $\int_{t_o}^{t_f} v\,dt$

distance = $\int_{\text{initial time}}^{\text{final time}} |v|\,dt$

$$\int_{t_o}^{t_f}\sqrt{(x')^2+(y')^2}\,dt \ *$$

average velocity =

$$= \frac{\text{final position} - \text{initial position}}{\text{total time}}$$

$$= \frac{\Delta x}{\Delta t}$$

BC TOPICS and important TRIG identities and values

l'Hôpital's Rule

If $\frac{f(a)}{g(b)} = \frac{0}{0}$ or $= \frac{\infty}{\infty}$,

then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Slope of a Parametric equation

Given a $x(t)$ and a $y(t)$ the slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Values of Trigonometric Functions for Common Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	"∞"
π	0	-1	0

Know both the inverse trig and the trig values. E.g. $\tan(\pi/4)=1$ & $\tan^{-1}(1)=\pi/4$

Euler's Method

If given that $\frac{dy}{dx} = f(x, y)$ and that the solution passes through (x_o, y_o),

$y(x_o) = y_o$

$\vdots$

$y(x_n) = y(x_{n-1}) + f(x_{n-1}, y_{n-1})\cdot\Delta x$

In other words:

$$x_{\text{new}} = x_{\text{old}} + \Delta x$$

$$y_{\text{new}} = y_{\text{old}} + \left.\frac{dy}{dx}\right|_{(x_{\text{old}}, y_{\text{old}})}\cdot\Delta x$$

Polar Curve

For a polar curve $r(\theta)$, the **AREA** inside a "leaf" is

$$\int_{\theta_1}^{\theta_2}\tfrac{1}{2}\left[r(\theta)\right]^2 d\theta$$

where θ_1 and θ_2 are the "first" two times that $r = 0$.

The **SLOPE** of $r(\theta)$ at a given θ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}\left[r(\theta)\sin\theta\right]}{\frac{d}{d\theta}\left[r(\theta)\cos\theta\right]}$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Integral of Log

Use IBP and let $u = \ln x$ (Recall u=LIPET)

$$\int \ln x\,dx = x\ln x - x + C$$

Ratio Test

The series $\sum_{k=0}^{\infty} a_k$ converges if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$$

If the limit equal 1, you know nothing.

Trig Identities

Double Argument

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

Pythagorean

$\sin^2 x + \cos^2 x = 1$

(others are easily derivable by dividing by $\sin^2 x$ or $\cos^2 x$)

$1 + \tan^2 x = \sec^2 x$

$\cot^2 x + 1 = \csc^2 x$

Reciprocal

$\sec x = \frac{1}{\cos x}$ *or* $\cos x\sec x = 1$

$\csc x = \frac{1}{\sin x}$ *or* $\sin x\csc x = 1$

Odd-Even

$\sin(-x) = -\sin x$ (odd)

$\cos(-x) = \cos x$ (even)

Some more handy INTEGRALS:

$$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Taylor Series

If the function f is "smooth" at $x = a$, then it can be approximated by the n^{th} degree polynomial

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Lagrange Error Bound

If $P_n(x)$ is the n^{th} degree Taylor polynomial of $f(x)$ about c and $\left|f^{(n+1)}(t)\right| \le M$ for all t between x and c, then

$$\left|f(x) - P_n(x)\right| \le \frac{M}{(n+1)!}\left|x-c\right|^{n+1}$$

Maclaurin Series

A Taylor Series about $x = 0$ is called Maclaurin.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Alternating Series Error Bound

If $S_N = \sum_{k=1}^{N}(-1)^n a_n$ is the N^{th} partial sum of a convergent alternating series, then

$$\left|S_\infty - S_N\right| \le \left|a_{N+1}\right|$$

Geometric Series

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$$

diverges if $|r| \ge 1$; converges to $\frac{a}{1-r}$ if $|r| < 1$

This is available at http://covenantchristian.org/bird/Smart/Calc1/StuffMUSTknowColdNew.htm

AP CALCULUS BC

Final Notes

Trigonometric Formulas

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$
4. $\sin(-\theta) = -\sin\theta$ ODD FUNCTION
5. $\cos(-\theta) = \cos\theta$ EVEN FUNCTION
6. $\tan(-\theta) = -\tan\theta$ ODD FUNCTION
7. $\sin(A+B) = \sin A\cos B + \sin B\cos A$
8. $\sin(A-B) = \sin A\cos B - \sin B\cos A$
9. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
10. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
11. $\sin 2\theta = 2\sin\theta\cos\theta$
12. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
13. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cot\theta}$
14. $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$
15. $\sec\theta = \dfrac{1}{\cos\theta}$
16. $\csc\theta = \dfrac{1}{\sin\theta}$
17. $\cos(\frac{\pi}{2} - \theta) = \sin\theta$
18. $\sin(\frac{\pi}{2} - \theta) = \cos\theta$

} COFUNCTION IDEA

Differentiation Formulas

1. $\dfrac{d}{dx}(x^n) = nx^{n-1}$
2. $\dfrac{d}{dx}(fg) = fg' + gf'$
3. $\dfrac{d}{dx}(\dfrac{f}{g}) = \dfrac{gf' - fg'}{g^2}$
4. $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
5. $\dfrac{d}{dx}(\sin x) = \cos x$
6. $\dfrac{d}{dx}(\cos x) = -\sin x$
7. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
8. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
9. $\dfrac{d}{dx}(\sec x) = \sec x\tan x$
10. $\dfrac{d}{dx}(\csc x) = -\csc x\cot x$
11. $\dfrac{d}{dx}(e^x) = e^x$
12. $\dfrac{d}{dx}(a^x) = a^x \ln a$
13. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
14. $\dfrac{d}{dx}(Arc\sin x) = \dfrac{1}{\sqrt{1-x^2}}$
15. $\dfrac{d}{dx}(Arc\tan x) = \dfrac{1}{1+x^2}$
16. $\dfrac{d}{dx}(Arc\sec x) = \dfrac{1}{|x|\sqrt{x^2-1}}$
17. $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ *Chain Rule*

Integration Formulas

1. $\int a\,dx = ax + C$

2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\quad n \neq -1$

3. $\int \frac{1}{x}\,dx = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$

6. $\int \ln x\,dx = x\ln x - x + C$

7. $\int \sin x\,dx = -\cos x + C$

8. $\int \cos x\,dx = \sin x + C$

9. $\int \tan x\,dx = \ln|\sec x| + C$ or $-\ln|\cos x| + C$

10. $\int \cot x\,dx = \ln|\sin x| + C$

11. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

12. $\int \csc x\,dx = \ln|\csc x - \cot x| + C = -\ln|\csc x + \cot x| + C$

13. $\int \sec^2 x\,dx = \tan x + C$

14. $\int \sec x \tan x\,dx = \sec x + C$

15. $\int \csc^2 x\,dx = -\cot x + C$

16. $\int \csc x \cot x\,dx = -\csc x + C$

17. $\int \tan^2 x\,dx = \tan x - x + C$

18. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} Arc\tan\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{\sqrt{a^2 - x^2}} = Arc\sin\left(\frac{x}{a}\right) + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} Arc\sec\frac{|x|}{a} + C = \frac{1}{a} Arc\cos\left|\frac{a}{x}\right| + C$

Formulas and Theorems

1. Limits and Continuity:

A function $y = f(x)$ is continuous at $x = a$ if

i). $f(a)$ exists

ii). $\lim_{x \to a} f(x)$ exists

iii). $\lim_{x \to a} = f(a)$

Otherwise, f is discontinuous at $x = 1$.

The limit $\lim_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal – that is,

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

2. Even and Odd Functions

1. A function $y = f(x)$ is even if $f(-x) = f(x)$ for every x in the function's domain. Every even function is symmetric about the y-axis.
2. A function $y = f(x)$ is odd if $f(-x) = -f(x)$ for every x in the function's domain. Every odd function is symmetric about the origin.

3. Periodicity

A function $f(x)$ is periodic with period $p\ (p > 0)$ if $f(x + p) = f(x)$ for every value of x.

Note: The period of the function $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ is $\frac{2\pi}{|B|}$.

The amplitude is $|A|$. The period of $y = \tan x$ is π.

4. Intermediate-Value Theorem

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

Note: If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a, b).

5. Limits of Rational Functions as $x \to \pm\infty$

i). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^2 - 2x}{x^3 + 3} = 0$

ii). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is infinite if the degrees of $f(x) >$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^3 + 2x}{x^2 - 8} = \infty$

iii). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{2x^2 - 3x + 2}{10x - 5x^2} = -\frac{2}{5}$

6. Horizontal and Vertical Asymptotes

1. A line $y = b$ is a horizontal asymptote of the graph $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \text{ or } \lim_{x \to -\infty} f(x) = b.$$

2. A line $x = a$ is a vertical asymptote of the graph $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \text{ or } \lim_{x \to a^-} = \pm\infty .$$

7. Average and Instantaneous Rate of Change

i). *Average Rate of Change*: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is $\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$.

ii). *Instantaneous Rate of Change*: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

8. Definition of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{of} \quad f'(\mathrm{a}) = \lim_{\mathrm{x} \to \mathrm{a}} \frac{f(\mathrm{x}) - f(\mathrm{a})}{\mathrm{x} - \mathrm{a}}$$

The latter definition of the derivative is the instantaneous rate of change of $f(\mathrm{x})$ with respect to x at $\mathrm{x} = \mathrm{a}$.
Geometrically, the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.

9. The Number ***e*** as a limit

i). $$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

ii). $$\lim_{n \to 0} \left(1 + \frac{n}{1}\right)^{\frac{1}{n}} = e$$

10. Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a, b) such that $f'(c) = 0$.

11. Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

12. Extreme-Value Theorem

If f is continuous on a closed interval $[a, b]$, then $f(x)$ has both a maximum and minimum on $[a, b]$.

13. To find the maximum and minimum values of a function $y = f(x)$, locate

1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist.
2. the end points, if any, on the domain of $f(x)$.

Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a minimum.

14. Let f be differentiable for $a < x < b$ and continuous for a $a \le x \le b$,

1. If $f'(x) > 0$ for every x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for every x in (a, b), then f is decreasing on $[a, b]$.

15. Suppose that $f''(x)$ exists on the interval (a,b)

1. If $f''(x) > 0$ in (a,b), then f is concave upward in (a,b).
2. If $f''(x) < 0$ in (a,b), then f is concave downward in (a,b).

To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ or where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

16a. If a function is differentiable at point $x = a$, it is continuous at that point. The converse is false, in other words, continuity does not imply differentiability.

16b. Local Linearity and Linear Approximations

The linear approximation to $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$ for x sufficiently close to x_0.

To estimate the slope of a graph at a point – just draw a tangent line to the graph at that point. Another way is (by using a graphing calculator) to "zoom in" around the point in question until the graph "looks" straight. This method almost always works. If we "zoom in" and the graph looks straight at a point, say $(a, f(a))$, then the function is locally linear at that point.

The graph of $y = |x|$ has a sharp corner at $x = 0$. This corner cannot be smoothed out by "zooming in" repeatedly. Consequently, the derivative of $|x|$ does not exist at $x = 0$, hence, is not locally linear at $x = 0$.

17. Dominance and Comparison of Rates of Change

Logarithm functions grow slower than any power function (x^n).

Among power functions, those with higher powers grow faster than those with lower powers.

All power functions grow slower than any exponential function $(a^x, a > 1)$.

Among exponential functions, those with larger bases grow faster than those with smaller bases.

We say, that as $x \to \infty$:

1. $f(x)$ grows faster than $g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(\ )} = \infty$ or if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

   If $f(x)$ grows faster than $g(x)$ as $x \to \infty$, then $g(x)$ grows slower than $f(x)$ as $x \to \infty$.

2. $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$ (L is finite and nonzero).

For example,

1. e^x grows faster than x^3 as $x \to \infty$ since $\lim_{x\to\infty} \frac{e^x}{x^3} = \infty$
2. x^4 grows faster than $\ln x$ as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^4}{\ln x} = \infty$
3. $x^2 + 2x$ grows at the same rate as x^2 as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^2 + 2x}{x^2} = 1$

To find some of these limits as $x \to \infty$, you may use the graphing calculator. Make sure that an appropriate viewing window is used.

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18. L'Hôpital's Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

19. Inverse function

1. If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f, then f and g are inverse functions of each other.
2. A function f has an inverse if and only if no horizontal line intersects its graph more than once.
3. If f is either increasing or decreasing in an interval, then f has an inverse.
4. If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and $g'(f(x)) = \frac{1}{f'(x)}$.

20. Properties of $y = e^x$

1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.
2. The domain is the set of all real numbers, $-\infty < x < \infty$.
3. The range is the set of all positive numbers, $y > 0$.
4. $\frac{d}{dx}(e^x) = e^x$
5. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
6. $y = e^x$ is continuous, increasing, and concave up for all x.
7. $\lim_{x \to +\infty} e^x = +\infty$ and $\lim_{x \to -\infty} e^x = 0$.
8. $e^{\ln x} = x$, for $x > 0$; $\ln(e^x) = x$ for all x.

21. Properties of $y = \ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.
2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.
3. $y = \ln x$ is continuous and increasing everywhere on its domain.
4. $\ln(ab) = \ln a + \ln b$.
5. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.
6. $\ln a^r = r \ln a$.
7. $y = \ln x < 0$ if $0 < x < 1$.
8. $\lim_{x \to +\infty} \ln x = +\infty$ and $\lim_{x \to 0^+} \ln x = -\infty$.
9. $\log_a x = \frac{\ln x}{\ln a}$

21. Trapezoidal Rule

If a function f is continuous on the closed interval $[a,b]$ where $[a,b]$ has been partitioned into n subintervals $[x_0,x_1], [x_1,x_2] \ldots [x_{n-1},x_n]$, each length $\frac{b-a}{n}$, then

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right].$$

22a. Definition of Definite Integral as the Limit of a Sum

Suppose that a function $f(x)$ is continuous on the closed interval $[a,b]$. Divide the interval into n equal subintervals, of length $\Delta x = \frac{b-a}{n}$. Choose one number in each subinterval, in other words, x_1 in the first, x_2 in the second, ..., x_k in the k th ,..., and x_n in the n th. Then

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx = F(b)-F(a).$$

22b. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

i). $\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c.

ii). $\int_a^a f(x)\,dx = 0$

iii). $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

iv). $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c.

v). If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

vi). If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

vii). If $f(x)\ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$

viii). If $g(x)\ge f(x)$ on $[a,b]$, then $\int_a^b g(x)\,dx \ge \int_a^b f(x)\,dx$

23. Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F'(x) = f(x), \text{ or } \frac{d}{dx}\int_a^b f(x)\,dx = f(x).$$

24. Second Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{or} \quad \frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(x)\cdot g'(x)$$

25. Velocity, Speed, and Acceleration

1. The velocity of an object tells how fast it is going and in which direction. Velocity is an instantaneous rate of change.
2. The speed of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.
   The speed of a particle increases (speeds up) when the velocity and acceleration have the same signs. The speed decreases (slows down) when the velocity and acceleration have opposite signs.
3. The acceleration is the instantaneous rate of change of velocity – it is the derivative of the velocity – that is, $a(t) = v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing. The acceleration gives the rate ot which the velocity is changing.

Therefore, if x is the displacement of a moving object and t is time, then:

i) velocity = $v(t) = x'(t) = \frac{dx}{dt}$

ii) acceleration = $a(t) = x''(t) = v'(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

iii) $v(t) = \int a(t)\,dt$

iv) $x(t) = \int v(t)\,dt$

Note: The average velocity of a particle over the time interval from t_0 to another time t, is

Average Velocity = $\frac{\text{Change in position}}{\text{Length of time}} = \frac{s(t)-s(t_0)}{t-t_0}$, where $s(t)$ is the position of the particle at time t.

26. The average value of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

27. Area Between Curves

If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a,b]$, then area between the curves is $\int_a^b [f(x) - g(x)]\,dx$.

~~28~~. Integration By "Parts"

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are continuous, then
$\int u\,dx = uv - \int v\,du$.

Note: The goal of the procedure is to choose u and dv so that $\int v\,du$ is easier to solve than the original problem.

Suggestion:

When "choosing" u, remember **L.I.A.T.E**, where **L** is the logarithmic function, **I** is an inverse trigonometric function, **A** is an algebraic function, **T** is a trigonometric function, and **E** is the exponential function. Just choose u as the first expression in **L.I.A.T.E** (and dv will be the remaining part of the integrand). For example, when integrating $\int x \ln x\,dx$, choose $u = \ln x$ since **L** comes first in **L.I.A.T.E**, and $dv = x\,dx$. When integrating $\int xe^{x}\,dx$, choose $u = x$, since x is an algebraic function, and **A** comes before **E** in **L.I.A.T.E**, and $dv = e^{x}dx$. One more example, when integrating $\int x\, Arc\tan(x)\,dx$, let $u = Arc\tan(x)$, since **I** comes before **A** in **L.I.A.T.E**, and $dv = x\,dx$.

29a. Volume of Solids of Revolution (rectangles drawn perpendicular to the axis of revolution)

Let f be nonnegative and continuous on $[a,b]$, and let R be the region bounded above by $y = f(x)$, below by the x-axis and the sides by the lines $x = a$ and $x = b$.

1. When this region R is revolved about the x-axis, it generates a solid (having circular cross sections) whose volume $V = \pi \int_a^b [f(x)]^2\,dx$.

2. When two functions are involved: $V = \pi \int_a^b \left(r_o^2 - r_i^2\right)\,dx$ where r_o is the distance between the axis of revolution and the furthest side of the shaded region and r_i is the distance between the axis of revolution and the nearest side of the shaded region.

3. When the rectangles are perpendicular to the x-axis, the integral will be in terms of x.
When the rectangles are perpendicular to the y-axis, the integral will be in terms of y.

29b. Volume of Solids with Known Cross Sections

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume = $\int_a^b A(x)\,dx$.

   Volumes on the interval [a, b] where $a(x)$ is the length of a side of the section:

   Square: $V = \int_a^b [a(x)]^2\,dx$ Equilateral Triangle: $V = \frac{\sqrt{3}}{4}\int_a^b [a(x)]^2\,dx$

   Semi-circle: $V = \frac{\pi}{8}\int_a^b [a(x)]^2\,dx$

   Isosceles Right Triangle: $V = \frac{1}{2}\int_a^b [a(x)]^2\,dx$ (when a = leg of triangle)

   Isosceles Right Triangle: $V = \frac{1}{4}\int_a^b [a(x)]^2\,dx$ (when a = hypotenuse of triangle)

2. For cross sections of area $A(y)$, taken perpendicular to the y-axis, volume = $\int_a^b A(y)\,dy$.

29c. Shell Method (rectangles drawn parallel to the axis of revolution)

1. Horizontal Axis of Revolution: $V = 2\pi\int_c^d p(y)h(y)\,dy$ (p is the distance between the axis of revolution and the center of a rectangle.)
2. Vertical Axis of Revolution: $V = 2\pi\int_a^b p(x)h(x)\,dx$ (p is the distance between the axis of revolution and the center of a rectangle.)

30. Solving Differential Equations: Graphically and Numerically

Slope Fields

At every point (x, y) a differential equation of the form $\frac{dy}{dx} = f(x, y)$ gives the slope of the member of the family of solutions that contains that point. A slope field is a graphical representation of this family of curves. At each point in the plane, a short segment is drawn whose slope is equal to the value of the derivative at that point. These segments are tangent to the solution's graph at the point.

The slope field allows you to sketch the graph of the solution curve even though you do not have its equation. This is done by starting at any point (usually the point given by the initial condition), and moving from one point to the next in the direction indicated by the segments of the slope field.

Some calculators have built in operations for drawing slope fields; for calculators without this feature there are programs available for drawing them.

X Euler's Method

Euler's Method is a way of approximating points on the solution of a differential equation $\frac{dy}{dx} = f(x, y)$. The calculation uses the tangent line approximation to move from one point to the next. That is, starting with the given point (x_1, y_1) – the initial condition, the point $(x_1 + \Delta x, y_1 + f'(x_1, y_1)\Delta x)$ approximates a nearby point on the solution graph. This aproximation may then be used as the starting point to calculate a third point and so on. The accuracy of the method decreases with large values of Δx. The error increases as each successive point is used to find the next. Calculator programs are available for doing this calculation.

31. Logistics

1. Rate is jointly proportional to its size and the difference between a fixed positive number (L) and its size.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \text{ which yields}$$

$$y = \frac{L}{1 + Ce^{-kt}} \text{ through separation of variables}$$

2. $\lim_{t\to\infty} y = L$; L = carrying capacity (Maximum); horizontal asymptote

3. y-coordinate of inflection point is $\frac{L}{2}$, i.e. when it is growing the fastest (or max rate).

32. Definition of Arc Length

If the function given by $y = f(x)$ represents a smooth curve on the interval $[a,b]$, then the arc length of f between a and b is given by $s = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$.

33. Improper Integral

$\int_a^b f(x)\, dx$ is an improper integral if

1. f becomes infinite at one or more points of the interval of integration, or
2. one or both of the limits of integration is infinite, or
3. both (1) and (2) hold.

34. Parametric Form of the Derivative

If a smooth curve **C** is given by the parametric equations $x = f(x)$ and $y = g(t)$, then the slope of the curve **C** at (x, y) is $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$, $\frac{dx}{dt} \neq 0$.

Note: The second derivative, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \div \frac{dx}{dt}$.

35. Arc Length in Parametric Form

If a smooth curve **C** is given by $x = f(t)$ and $y = g(t)$ and these functions have continuous first derivatives with respect to t for $a \le t \le b$, and if the point $P(x, y)$ traces the curve exactly once as t moves from $t = a$ to $t = b$, then the length of the curve is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

$$\text{speed} = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

36. Polar Coordinates

1. Cartesian vs. Polar Coordinates. The polar coordinates (r, θ) are related to the Cartesian coordinates (x, y) as follows:

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \text{ and } x^2 + y^2 = r^2$$

2. To find the points of intersection of two polar curves, find (r, θ) satisfying the first equation for which some points $(r, \theta + 2n\pi)$ or $(-r, \theta + \pi + 2n\pi)$ satisfy the second equation. Check separately to see if the origin lies on both curves, i.e. if r can be 0. Sketch the curves.

Area in Polar Coordinates: If f is continuous and nonnegative on the interval $[\alpha, \beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

Derivative of Polar function: Given $r = f(\theta)$, to find the derivative, use parametric equations.

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

$$\text{Then} \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

Arc Length in Polar Form: $s = \int_{\alpha}^{\beta}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

Sequences and Series

1. If a sequence $\{a_n\}$ has a limit L, that is, $\lim_{n\to\infty} a_n = L$, then the sequence is said to converge to L. If there is no limit, the series diverges. If the sequence $\{a_n\}$ converges, then its limit is unique. Keep in mind that

   $\lim_{n\to\infty}\frac{\ln n}{n} = 0;\quad \lim_{n\to\infty} x^{\left(\frac{1}{n}\right)} = 1;\quad \lim_{n\to\infty}\sqrt[n]{n} = 1;\quad \lim_{n\to\infty}\frac{x^n}{n!} = 0$. These limits are useful and arise frequently.

2. The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges; the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$ and $a \neq 0$.

3. The p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

4. Limit Comparison Test: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of nonnegative terms, with $a_n \neq 0$ for all sufficiently large n, and suppose that $\lim_{n\to\infty}\frac{b_n}{a_n} = c > 0$. Then the two series either both converge or both diverge.

5. Alternating Series: Let $\sum_{n=1}^{\infty} a_n$ be a series such that
   - i) the series is alternating
   - ii) $|a_{n+1}| \leq |a_n|$ for all n, and
   - iii) $\lim_{n\to\infty} a_n = 0$

   Then the series *converges*.

Alternating Series Remainder: The remainder R_N is less than (or equal to) the first neglected term

$$|R_N| \le a_{N+1}$$

6. The *n*-th Term Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.
   Note that the converse is *false*, that is, if $\lim_{n\to\infty} a_n = 0$, the series may or may not converge.

7. A series $\sum a_n$ is absolutely convergent if the series $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ does not converge, then the series is conditionally convergent. Keep in mind that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

8. Comparison Test: If $0 \le a_n \le b_n$ for all sufficiently large n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

9. Integral Test: If $f(x)$ is a positive, continuous, and decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ will converge if the improper integral $\int_1^{\infty} f(x)\,dx$ converges. If the improper integral $\int_1^{\infty} f(x)\,dx$ diverges, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

10. Ratio Test: Let $\sum a_n$ be a series with nonzero terms.

    i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges absolutely.

    ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series is divergent.

    iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive (and another test must be used).

11. Power Series: A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots \quad \text{or}$$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots + c_n (x-a)^n + \ldots$$ in which the

center a and the coefficients $c_0, c_1, c_2, ..., c_n, ...$ are constants. The set of all numbers x for which the power series converges is called the interval of convergence.

12. Taylor Series: Let f be a function with derivatives of all orders throughout some intervale containing a as an interior point. Then the Taylor series generated by f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

The remaining terms after the term containing the nth derivative can be expressed as a remainder to Taylor's Theorem:

$$f(x) = f(a) + \sum_{1}^{n} f^{(n)}(a)(x-a)^n + R_n(x) \text{ where } R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Lagrange's form of the remainder: $R_n x = \dfrac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$, where $a < c < x$.

The series will converge for all values of x for which the remainder approaches zero as $x \to \infty$.

13. Frequently Used Series and their Interval of Convergence

$$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + ... = \sum_{n=0}^{\infty} x^n, \ |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \ |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \ |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^n \frac{x^{2n}}{(2n)!} + ... = \sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!}, \ |x| < \infty$$

AP Calculus – Final Review Sheet

When you see the words .... **This is what you think of doing**

1. Find the zeros of a function.	Set the function equal to zero and solve for x.
2. Find equation of the line tangent to f(x) at (a,f(a)).	Find f '(x),the derivative of f(x). Evaluate f '(a). Use the point and the slope to write the equation: y= f '(a)(x-a)+f(a)
3. Find equation of the line normal to f(x) at (a,f(a)).	Find f '(x),the derivative of f(x). Evaluate f '(a). The slope of the normal line is $-\frac{1}{f'(a)}$. Use the point and the slope to write the equation: $y = \frac{1}{f'(a)}(x-a)+f(a)$
4. Show that f(x) is even.	Evaluate f at x = -a and x = a and show they are equal.
5. Show that f(x) is odd.	Evaluate f at x = -a and x = a and show they are opposite.
6. Find the interval where f(x) is increasing.	Find f '(x) and find all intervals in the domain of f and f ' where f '(x) > 0.
7. Find the interval where the slope of f(x) is increasing.	Find f "(x) and find all intervals in the domain of f, f ', and f " where f "(x) >0.
8. Find the relative minimum value of a function f(x).	Find all the critical points for f, where f '(x)=0 or f '(x) does not exist. Find all locations where f ' changes from negative to positive or where f changes from decreasing to increasing.
9. Find the absolute minimum slope of a function f(x) on [a,b].	Find all critical points of f ', where f "(x)=0 or f " (x) does not exist. Evaluate f '(x) at all critical points of f ' and the endpoints. From these values find where f ' is minimum.
10. Find critical values for a function f(x).	Find f '(x) and then locate all points where f '(x)=0 or f '(x) does not exists.
11. Find inflection points of a function f(x).	Find f "(x) and then find all locations where f "(x) changes sign.
12. Show that $\lim_{x \to a} f(x)$ exists.	Find $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ and show they are equal.
13. Show that f(x) is continuous.	For each point in the domain a, find f(a), and $\lim_{x \to a} f(x)$. Show that $\lim_{x \to a} f(x) = f(a)$.
14. Find vertical asymptotes of a function f(x).	Look at the definition of the function f(x). If f is written in a ratio, first check that the function cannot be simplified. Then locate all places where the denominator of the function equals zero.
15. Find horizontal asymptotes of function f(x).	Find $\lim_{x \to +\infty} f(x) = k_1$ and $\lim_{x \to -\infty} f(x) = k_2$. Each of these values if an answer to a horizontal asymptote: $y = k_1$ and $y = k_2$.

16. Find the average rate of change of f(x) on [a,b].	This is the slope of the secant line between (a,f(a)) and (b, f(b)) or $\frac{f(b)-f(a)}{b-a}$.
17. Find instantaneous rate of change of f(x) on [a,b].	This is another name for f '(a), or the derivative the function evaluated at x = a.
18. Find the average value of f(x) on [a,b].	This means to find the average value that f takes on between (a, f(a)) and (b, f(b)). It is found by find the area of the function bounded by x=a. x=b, x=0, and y=f(x). Then divide this by the width of the interval b-a. It is written as $\frac{\int_a^b f(x)dx}{b-a}$.
19. Find the absolute maximum of f(x) on [a,b].	Find all the critical points for f, where f '(x)=0 or f '(x) does not exist. Evaluate the function at all critical points of f and endpoints. From these values find where f is maximum.
20. Show that a piecewise function is differentiable at the point *a* where the function rule splits	Find the derivative of each piece of the function. Show that the $\lim_{x\to a} f'(x)$ exists or is equal from the left and the right.
21. Given s(t), the position function, find v(t), the velocity function.	Find the derivative of s(t).
22. Given v(t), the velocity function, find how far a particle travels on [a,b].	$\int_a^b \lvert v(t)\rvert dt$. Remember that $\int_a^b v(t)dt$ only find the net distance traveled.
23. Find the average velocity of a particle on [a,b] given s(t), the position function. Find the average velocity of a particle on [a,b] given v(t), the velocity function.	This is the slope of the secant line: $\frac{s(b)-s(a)}{b-a}$. The second one is the average value of the function or $\frac{\int_a^b v(t)dt}{b-a}$.
24. Given v(t), the velocity function, determine the intervals where a particle is speeding up.	Evaluate v(t) for its sign. Find the derivative of v(t) to determine a(t). Determine when the particle in stationery (v(t)=0). Determine when a(t)=0. Study the intervals where the particle is initially at rest and then shows positive or negative velocity, which means it will move left or right. The particle will have to speed up until it reaches point where a(t)=0. Locate the point where the particle will have an a(t)=0. (Now it will begin to slow down and eventually come to rest again.
25. Given v(t), the velocity function, and s(0), the initial position, find s(t), the position function.	$s(t) = s(0) + \int_0^t v(x)dx$
26. Show that Rolle's Theorem holds for a function f(x) on [a,b].	Verify that f(x) is continuous on [a,b] and differentiable on (a,b). Verify that f(a)=0 and f(b)=0. Then you are guaranteed that there exists a point c (a<c<b) where f '(c)=0.

27. Show that the Mean Value Theorem holds for a function f(x) on [a,b].	Verify that f(x) is continuous on [a,b] and differentiable on (a,b). Then you are guaranteed that there exists a point c (a<c<b) where $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.
28. Find domain of f(x).	Analyze the function f. Look for radical expressions in the description. Determine values of x that cannot be used within the radical. Exclude these from the domain. Look at the denominator. If the denominator contains a polynomial, find the zeros for this polynomial and exclude these x values from the domain.
29. Find range of f(x) on [a,b].	If f is continuous on [a,b], then the range of f will between [minimum value of f, maximum value of f].
30. Find range of f(x) on $(-\infty,\infty)$.	If f is continuous on $(-\infty,\infty)$ then you will need to consider $\lim\limits_{x\to+\infty} f(x)=k_1$ and $\lim\limits_{x\to-\infty} f(x)=k_2$. If these limits are above the local maximum or below the local minimum the range will be $[k_1,k_2]$. Otherwise you will have to adjust the range. If the limits go to infinity then the range is $(-\infty,\infty)$.
31. Find f ' (x), the derivative of f(x), by definition	Use $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$
32. Given two functions f and f^{-1} are inverse functions (f(a)=b and f^{-1}(b)=a) and f '(a), find derivative of inverse function f^{-1} at x=b.	$\left(f^{-1}\right)'(b)=\dfrac{1}{f'(a)}$
33. Given $\dfrac{dy}{dt}$ is increasing proportionally to *y*, find a family of functions that describe the population as a function of time.	$\dfrac{dy}{dt}=ky$ then separate the variables, integrate each side and add a constant of integration to one side.
34. Find the line x=c that divides the area under f(x) on [a,b] to two equal areas	Find a point c such that $\int_a^c f(x)dx=\dfrac{\int_a^b f(x)dx}{2}$
35. $\dfrac{d}{dx}\int_a^x f(t)\,dt=$	f(x)
36. Given that u is some function of x find $\dfrac{d}{dx}\int_a^u f(u)\,dt=$	$f(u)\dfrac{du}{dx}$

37. Find the area bounded by f(x), the x-axis, x=1 and x = 10 using 3 trapezoids, where $\Delta x=3$.

Find f(1), f(4), f(7), and f(10). Use these for the bases in finding the area of three trapezoids with heights of 3:

$$\frac{1}{2}(3)(f(1)+f(4))+\frac{1}{2}(3)(f(4)+f(7))+\frac{1}{2}(3)(f(7)+f(10))$$

38. Approximate the area bounded by f(x), the x-axis, x=0 and x = 7 using left Reimann sums from information about f(x) given in tabular data.

x	0	1	5	7
f(x)	1	13	16	5

Find the base, difference between x values, and height (at left hand end) of the three rectangles.

$$(1)(1)+(4)(13)+(2)(16)$$

39. Approximate the area bounded by f(x), the x-axis, x=0 and x = 7 using right Reimann sums from information about f(x) given in tabular data.

x	0	1	6	7
f(x)	-1	-13	-16	-5

Find the base, difference between x values, and height (at right hand end) of the three rectangles.

$$(1)(-13)+(5)(-16)+(1)(-5)$$

41. Approximate the area bounded by f(x), the x-axis, x = 0, and x = 14 using two subintervals and midpoint rectangles from information about f(x) given in tabular data.

x	0	3	6	10	14
f(x)	1	7	12	11	3

Find the intervals for the two rectangles: (0,6) and (6,14). The midpoints are 3 and 10. Find the height of the rectangles: 7 and 11 respectively. Find the area: $(6)(7)+(8)(11)$

40. Approximate the area bounded by f(x), the x-axis, x = 0, and x = using three trapezoids from information about f(x) given in tabular data.

x	1	5	6	10
f(x)	2	7	12	15

Find the height of the three trapezoids: 4, 1, and 4. Find the bases: 2 and 7, 7 and 12, and 12 and 15. Find the areas:

$$\frac{1}{2}(4)(2+7)+\frac{1}{2}(1)(7+12)+\frac{1}{2}(4)(12+15)$$

42. Given the graph of f ' (x) >0 between x=0 and x = a and f(0) =8, find f(a).

$f(a)=f(0)+\int_0^a f'(x)dx$ So to find the integral you can find the area under the f ' graph between x=0 and x=a.

43. Solve the differential equation $\frac{dy}{dx}=\frac{1+x}{y}$.

Separate the variables and then integrate each side. Remember to include a constant of integration. If possible find the constant through substitution.

44. Describe the meaning of $\int_a^x f(t)\,dt$

Suppose f(x) is a rate equation for F(t). Then this integral represent the net change in F(t) from time a to time x.

45. Given a base is bounded by x = a, x = b, f(x) and g(x), where f(x) < g(x) for all a<x<b, find the volume of the solid whose cross section, perpendicular to the x-axis are squares.

Volume of the solid = $\int_a^b (g(x)-f(x))^2\,dx$

46. Find where the tangent line to f(x) is horizontal.

Find f '(x) and then set f '(x) =0 and solve for x.

47. Find where the tangent line to f(x) is vertical.

Find f '(x) and then analyze f '(x) to determine where f '(x) is undefined because of a denominator.

48. Find the minimum acceleration given v(t), the velocity function.	Find a(t) or the derivative of v(t) and a'(t). Find the critical points for a(t) from a'(t). Find where a'(t) is changing from negative to positive (a(t) changing from decreasing to increasing). These are locations for the local minimum accelerations.
49. Approximate the value of f(1.1) by using the tangent line to *f* at x=1.	Write the tangent line at x=1. $y = f'(1)(x-1) + f(1)$. Use x = 1.1 in this tangent line to find the approximate value of f(1.1).
50. Given the value of F(a) and the fact that the anti-derivative of *f* is *F*, find F(b).	$F(b) = F(a) + \int_a^b f(x)dx$
51. Find the derivative of f(g(x)).	$(f(g(x)))' = f'(g(x)) \cdot g'(x)$
52. Given $\int_a^b f(x)dx$, find $\int_a^b [f(x)+k]dx$	$\int_a^b [f(x)+k]\,dx = \int_a^b f(x)dx + \int_a^b kdx =$ $\int_a^b f(x)dx + k(b-a)$
53. Given a graph of f '(x), find where f(x) is increasing.	From the graph of f '(x) find where the graph is below the x-axis. This means f '(x) is negative. Describe these intervals.
54. Given v(t), the velocity function, and s(0), the initial position, find the greatest distance from the origin of a particle on [0,b].	Find when v(t) is zero. This means the function is at rest at these values. Write s(t). $s(t) = s(0) + \int_0^t v(x)dx$. Evaluate s(t) at each place v(t) is zero. Pick out the greatest distance from the origin.
55. Given a water tank with *g* gallons initially, is being filled at the rate of F(t) gallons/min and emptied at the rate of E(t) gallons/min on $[t_1, t_2]$, find the amount of water in the tank at *m* minutes where $t_1 < m < t_2$.	$\int_{t_1}^{m} (F(t) - E(t))\,dt$
56. Given a water tank with *g* gallons initially, is being filled at the rate of F(t) gallons/min and emptied at the rate of E(t) gallons/min on $[t_1, t_2]$, find the rate the water amount is changing at *m*.	F(t)-E(t)
57. Given a water tank with *g* gallons initially, is being filled at the rate of F(t) gallons/min and emptied at the rate of E(t) gallons/min on $[t_1, t_2]$, find the time when the water is at a minimum.	Differentiate the integral in question 55 with respect to t. This will give you a rate equation or the equation in question 56. Find the zeros for F(t)-E(t). Evaluate the integral from question 55 at these zero's and the endpoints. Pick out the minimum value.
58. Given a chart of *x* and f(x) on selected values between *a* and *b*, estimate f '(c) where *c* is between *a* and b.	Use two sets of points (a, f(a)) and (b, f(b)) near c to evaluate $f'(c) \approx \dfrac{f(b) - f(a)}{b - a}$

59. Given $\frac{dy}{dx}$, draw a slope field	Identify points on the graph. Name the coordinates of these points. Evaluate $\frac{dy}{dx}$ at these points. Draw a short line that represents the given slope at that point. The slope field should model the slope of a family of functions whose derivative is $\frac{dy}{dx}$.
60. Given that f(x) < g(x). find the area between curves f(x) and g(x) between x = a and x = b on [a,b].	$\int_a^b (f(x)-g(x))dx$
61. Given that f(x) > g(x). Find the volume of the solid created if the region between curves f(x) and g(x) between x = a and x = b on [a,b]. is revolved about the x-axis.	$\pi\int_a^b \left((f(x))^2-(g(x))^2\right)dx$
62. Find a limit in the form $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.	Determine the value of a and the function f. Differentiate f and evaluate at a.
63. Given information about f(x) for x in [a,b], show that there exists a c in the interval [a,b], where $f'(c) = \frac{f(b)-f(a)}{b-a}$.	Check to see that f(x) is continuous on [a,b] and differentiable on (a,b). Then the Mean Value Theorem guarantees that there exists a c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
64. Given f " (x) and all critical values of x in (a,b) where f '(x)=0, determine the location of all relative extrema for f.	Check the concavity of f at each critical value where f '(x) = 0. If f"(x)>0 you have found the location of a minimum. If f"(x)<0 you have found the location of a maximum.
65. Given f '(x) in graphical form on a domain (a,b), determine the location of all relative extrema for f.	Find locations where the graph of f ' is changing from being below the x-axis to being above the x-axis. This is a location of a relative minimum. Find locations where the graph of f ' is changing from being above the x-axis to being below the x-axis. This is a location of a relative maximum.
66. Given that functions f and g are twice differentiable, find h '(x) if h(x) = f(x)g(x) +k.	h '(x)=f(x)g '(x) +g(x)f '(x)

Things To Remember

(Common Mistakes That Make Readers Pull Their Hair Out.)

"Ben Cornelius from the Oregon Institute of Technology compiled this list several years ago. It still works for me and my students." from AP Calc listserv April 27,2005

1. There is no need to simplify arithmetic. It won't make the answer any more correct (even in a long Riemann sum).

2. Don't cross out your work unless you know you can do better.

3. Be sure to label your answers and use correct units.

4. If you are worried that your result in part (a) is incorrect, use it anyway to finish the problem.

5. If you use your calculator, describe it clearly in mathematical terms, not in calculator speak.

6. Don't write bad math. (e.g. "Slope of the derivative." or "6.2368 = 6.237" or "-17.21 = 17.21")

7. Remember: 3 decimal places, rounded or truncated. (More is ok.)

8. Don't write $f(x) = 2(1.5) + 3$ when you really mean $f(1.5) = 2(1.5) + 3$.

9. Every pronoun needs an antecedent. Name the function you are referring to. Do not say, "The slope is ...". Say, "The slope of g is ....", especially when more than one function is being discussed.

10. When asked to write an integral, start with the limits and any constants of multiplication. Then you can make a guess as to the integrand.

11. Know the difference between increasing and positive. f is increasing when f' is positive.

12. Calculator work will be limited to the four required functionalities: graphing, roots, numerical derivative, and numerical integration. You will not be required to do anything else with your calculator and no question will be asked where using an additional feature would give an advantage. (e.g. curve fitting)

13. Know the difference between local and global extrema.

14. Know the difference between the extreme value (y-coordinate) and the location of the extreme value (x- and y-coordinate).

15. When justifying local extrema or points of inflection, make sure your number line or chart is labeled. Summarize the results in complete sentences.

AP Calculus Exam Date and Time: Monday, May 13, 2023; AM Session

Format: There are two sections of this exam given in 3 hours and 15 minutes. Both sections count equally towards your final grade and both sections cover the full range of topics.

- Section I Part A (60 minutes) 30 multiple-choice questions for which you may NOT use a calculator.

- Section I Part B (45 minutes) 15 multiple-choice questions. You may use your calculator in this part. Some of these questions require the use of a graphing calculator and others do not.

For both parts of Section I (multiple-choice) you receive one point for each correct answer. There is no deduction for an incorrect answer or for a question that is left blank. You may not return to Section I Part A if you complete this part early

IMPORTANT: The multiple-choice questions are machine scored. YOUR WORK WILL NOT BE GRADED IN THESE SECTIONS.

- Section Il Part A (30 minutes) 2 free-response questions. You may use your calculator in this part. At least one question will require calculator use. In this part you will find longer questions with several related parts. You are required to show your work, Be sure to write clearly and legibly. If you make an error, you may cross it out. Erased or crossed-out work will not be graded (do not use White-Out). Clearly indicate the methods you use, because you will be graded on the correctness of your methods as well as the accuracy of your final answers. Answers without work lose credit, Justifications require mathematical (non-calculator) reasons.

- Section Il Part B (60 minutes) 4 free-response questions. You may NOT use your calculator in this part. Again, you will find longer multi-part questions in which you are required to show your work. You may return to Section 11 Part A WITHOUT A CALCULATOR if time permits.

Calculator Requirements and Tips: Know how to do these 4 things on your calculator;

1. Graph a function in any window. Know how to change the viewing window to fit the function.
2. Solve an equation. There are several ways to do this. You may set the equation equal to zero, graph and find the root(s). Or, you may graph both sides of the equation separately and find the point(s) of intersection.
3. Find the numerical value of the derivative of a function at a given point (using Math 8).
4. Find the numerical value of a definite integral (using Math 9).

- Be sure your calculator is in radian mode.

- Your work must be expressed in mathematical notation rather than calculator syntax.

- Numerical answers may be left unsimplified. If you do give a decimal approximation, it must be correct to three places after the decimal point. Do not round off too soon, Use the STORE feature for working with intermediate results.

When finding min/max and inflections points, if you use a number-line schematic, be sure to write, in words, any justifications for your answers. Identify the functions by name (for f 'or f") and be sure to indicate which functions are positive/negative, which functions increase/decrease, and which functions are concave up/concave down. DO NOT USE PRONOUNS when writing these justifications. If you use the Candidate Test for absolute extrema, show your table of values with the headings labeled properly.

Common Free-Response Mistakes

- Missing limits of integration.
- Missing the independent variable designation in an integral (dx or dy).
- Missing the constant of integration.
- Not considering the endpoints of an interval (for example, when looking for the absolute maximum value of a function).
- Giving an answer from points outside the given interval.
- Not giving both coordinates of a point when required.
- Giving both coordinates when only one is asked for. Remember, "value of a function" means the y-value.
- Having the calculator in degrees.
- Not answering the question which was asked even though all the work is correct.
- Using "=" when $\approx$ is required (or vice versa).
- Ignoring units of measure.
- Using calculator features or programs Other than the basic four skills without mathematical justification.
- Curve fitting: Occasionally a function is given as a graph or a table of values. You are being asked to demonstrate that you can work from the graphical or numerical data. The questions which accompany the data are meant to be answered without an equation. You may have learned how to approximate functions using various curve-fitting operations (regressions) built into your calculator. This is not allowed. Using an approximate equation of the given data will not earn credit. You must only work with the given function, i.e. the given table of values.
- Unclear writing — using pronouns or incomplete labels when writing about functions and their derivatives.

What to Bring:

- Several sharpened pencils with erasers for multiple choice.
- Several blue or black pens.
- Your graphing calculator with extra batteries (scientific calculators are not allowed).
- A watch (that doesn't make any noise or does any tricks beyond telling time).
- A well-rested body and an alert mind!!

- Cell phones, computers, or any other electronic or communication device.
- Books, compasses, colored pencils, correction fluid, dictionaries, highlighters, scrap paper or notes.
- Rulers and straight-edges.
- A calculator with a QWERTY keyboard.
- Food or drink.
- Clothing with subject-related information.

What NOT to Bring: