Do Now: #4 from 2016 Free Response

$\frac{dy}{dx}$ und. when $x=1,\ y\neq 0$

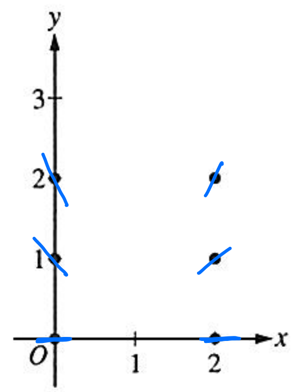
2016 AP® CALCULUS AB FREE-RESPONSE QUESTIONS

4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

$\frac{dy}{dx}=0$ when $y=0,\ x\neq 1$

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(x,y)	$\frac{dy}{dx}$
$(0,1)$	$\frac{1}{-1}=-1$
$(0,2)$	$\frac{2^2}{0-1}=-4$
$(2,1)$	$\frac{1}{2-1}=1$
$(2,2)$	$\frac{4}{2-1}=4$



(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

(b) $(2,3)$ $\quad \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3^2}{2-1} = 9$

$$y-3 = 9(x-2)$$
$$y = 9(x-2)+3$$
$$y = 9(2.1-2)+3 = 3.9$$

(c) $\int \frac{dy}{y^2} = \int \frac{dx}{x-1}$

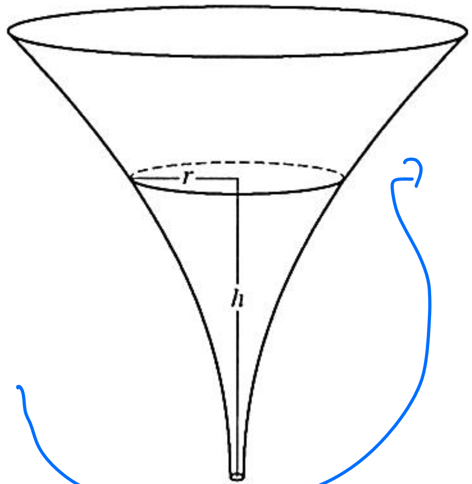
$$\int y^{-2}\,dy = \int \frac{dx}{x-1}$$
$$\frac{y^{-1}}{-1} = \ln|x-1| + C$$
$$-\frac{1}{y} = \ln|x-1| + C$$
$$-\frac{1}{3} = \ln|2-1| + C$$

$(2,3)$

$\to C = -\frac{1}{3}$

$$-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$$
$$\frac{1}{y} = -\ln|x-1| + \frac{1}{3}$$
$$y = \frac{1}{-\ln|x-1| + \frac{1}{3}}$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$



a) $\frac{1}{10-0}\int_0^{10} \frac{1}{20}(3+h^2)\,dh$

$\frac{1}{10}\cdot\frac{1}{20}\left[3h+\frac{h^3}{3}\Big|_0^{10}\right]$

$\frac{1}{200}\left[30+\frac{1000}{3}-0\right]$ in

5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}(3+h^2)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

(b) $A_0 = \pi r^2$ $\quad V = \int_0^{10} \pi\left(\frac{1}{20}(3+h^2)\right)^2 dh$

$V = \frac{\pi}{400}\left[\int_0^{10}(9+6h^2+h^4)\,dh\right]$

$V = \frac{\pi}{400}\left[9h+2h^3+\frac{h^5}{5}\Big|_0^{10}\right]$

$\frac{\pi}{400}\left[90+2(10)^3+\frac{10^5}{5}-0\right]$ in^3

c)

$\frac{dr}{dt} = -\frac{1}{5}$ in/s $\qquad h = 3$ in

$$r = \frac{1}{20}(3 + h^2)$$

$$\frac{dr}{dt} = \frac{1}{20}\left(2h\frac{dh}{dt}\right)$$

$$-\frac{1}{5} = \frac{1}{20}\left(2(3)\frac{dh}{dt}\right)$$

$$-\frac{1}{5} = \frac{6}{20}\frac{dh}{dt}$$

$$-\frac{1}{\cancel{5}} \cdot \frac{\cancel{20}^{4}}{6} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-4}{6} \text{ in/s}$$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	−6	3	2	8
2	2	−2	−3	0
3	8	7	6	2
6	4	5	3	−1

6. The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

(b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

$u = 2x$
$du = 2dx$
$\frac{1}{2}du = dx$

$\frac{1}{2}\int_2^6 f''(u)\,du = \frac{1}{2}\left[f'(6) - f'(2)\right]$

$\frac{1}{2}\left[5 - (-2)\right]$

(a) $k(3) = f(g(3)) = f(6) = 4$ $\quad (3,4)$

$k'(x) = f'(g(x)) \cdot g'(x)$

$k'(3) = f'(\overset{6}{g(3)}) \cdot g'(3) = 5 \cdot 2 = 10$

$y - 4 = 10(x-3)$

(b) $h'(x) = \dfrac{f(x)g'(x) - g(x)f'(x)}{f^2(x)}$

$h'(1) = \dfrac{f(1)g'(1) - g(1)f'(1)}{f^2(1)} = \dfrac{(-6)(8) - (2)(3)}{(-6)^2}$

given

Homework 05-01

(4) (a) $(0, 91)$ → point

slope → $\left.\frac{dH}{dt}\right|_{t=0} = -\frac{1}{4}(91-27) = -16$

$y - 91 = -16t$ → eq of tan line

$y - 91 = -16(3)$

$y = -48 + 91 = 43°C$

(b) [look at concavity of H]

$\frac{d^2H}{dt^2} = -\frac{1}{4} \cdot \frac{dh}{dt}$

$= -\frac{1}{4} \cdot -\frac{1}{4}(H-27)$ [given] in the problem

$\left.\frac{d^2H}{dt^2}\right|_{t=3} > 0$ because $H > 27$ for $t > 0$

Since $\frac{d^2H}{dt^2} > 0$ when $t = 3$, then H is concave up so the tan line approx from part (a) is an underestimate.

(c) [integrate deriv to get G]

$$\frac{dG}{dt} = -(G-27)^{2/3}$$

$$dG = -(G-27)^{2/3}\,dt$$

$$\frac{dG}{-(G-27)^{2/3}} = dt$$

[separate variables first]

$$-\int (G-27)^{-2/3}\,dG = \int dt$$

$$-\frac{G-27^{1/3}}{\frac{1}{3}} = t + C$$

$$-3(G-27)^{1/3} = t + C$$

$$-3(91-27)^{1/3} = 0 + C$$

$$-3(\sqrt[3]{64}) = C = -12$$

[use $G(0)=91$ to find C]

$$-3(G-27)^{1/3} = t - 12$$

$$(G-27)^{1/3} = -\tfrac{1}{3}t + 4$$

$$G - 27 = \left(-\tfrac{1}{3}t + 4\right)^3$$

$$G(t) = \left(-\tfrac{1}{3}t + 4\right)^3 + 27$$

[isolate G(t)]

$$G(3) = \left(\left(-\tfrac{1}{3}\cdot 3 + 4\right)^3 + 27\right)^\circ C$$

or

$$= 54^\circ C$$

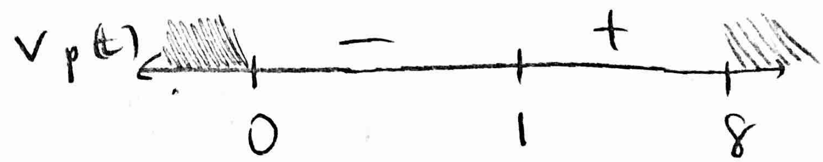
[does not need to be simplified but need units]

(5) (a) [need $v_p(t)$, which is $x_p'(t) < 0$]

$$v_p(t) = \frac{1}{t^2-2t+10} \cdot (2t-2)$$

$$= \frac{2(t-1)}{t^2-2t+10}$$

$v_p(t) = 0$ when $t = 1$



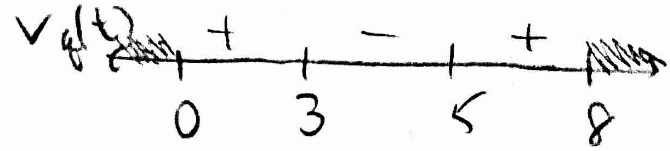
include since given $0 \le t \le 8$

P moving left when $0 \le t < 1$ because $v_p(t) < 0$.

(b) [need $v_p(t)$ and $v_q(t)$ to have same signs.]

$$v_q(t) = t^2 - 8t + 15 = (t-3)(t-5)$$

$v_q(t) = 0$ when $t = 3, 5$



[look at # lines from part (a) & part (b).]

P & Q travel in same direction when $1 < t < 3$ and $5 < t \le 8$ because $v_p(t)$ & $v_q(t)$ have same sign.

(c) $V_q'(t) = a_q(t) = 2t - 8$

$a_q(2) = -4$ [Remember to look at signs of v & a to know if speed is inc or dec]

At $t=2$, the speed of Q is decreasing because $a_q(2)$ & $V_q(2)$ have opposite signs.

(d) [integrate v_q to get position of q, $x_q(t)$]

$\int_0^c (t^2 - 8t + 15)\,dt$ → c is first time Q changes direction

$= x_q(c) - x_q(0)$ [given as 5]

[Need to find c → look at $v_q(t)$ to see where changes sign for first time - can use # line from part (b)]

$c = 3$ because this is the first value where $v_q(t)$ changes sign

$$\int_0^3 (t^2 - 8t + 15)\,dt = \left(\frac{t^3}{3} - \frac{8t^2}{2} + 15t\right)\Big]_0^3$$

$$= \left(\frac{1}{3}t^3 - 4t^2 + 15t\right)\Big]_0^3$$

$$= (9 - 36 + 45) - 0$$

$$9 - 36 + 45 = x_q(3) - 5$$

$$9 - 36 + 45 + 5 = x_q(3)$$

or

23

[does not need to be simplified]

(6) (a) $f'(x) = 2 \cdot -\sin(2x) + e^{\sin x} \cdot \cos x$

$f'(\pi) = -2\sin 2\pi + e^{\sin \pi} \cdot \cos \pi$

$= -2(0) + e^{0} \cdot -1$

$= -1$ → slope of tan line at $x = \pi$

(b) $k'(x) = h'(f(x)) \cdot f'(x)$

$k'(\pi) = h'(f(\pi)) \cdot \underbrace{f'(\pi)}_{\text{from part a}}$

$= h'(\cos 2\pi + e^{\sin \pi}) \cdot -1$

$= h'(1 + e^{0}) \cdot -1$

$= h'(1+1) \cdot -1$

$= \underbrace{h'(2)}_{\text{slope of h when } x=2} \cdot -1 = -\frac{1}{3}(-1) = \frac{1}{3}$

(c) $m'(x) = g(-2x) \cdot h'(x) + h(x) \cdot g'(-2x) \cdot -2$

$m'(2) = g(-4) \cdot h'(2) + h(2) \cdot g'(-4) \cdot -2$

$= \underbrace{5}_{\text{from table}} \cdot \underbrace{-\frac{1}{3}}_{\text{slope of h when } x=2} + \underbrace{-\frac{1}{3}(2)}_{\text{from graph of h } y=-\frac{1}{3}x \text{ when } 0<x<3} \cdot \underbrace{-1}_{\text{from table}} \cdot -2$

[you do not need to simplify]

OR

$= -3$

(d) Since g is differentiable, g is also continuous on $[5,-3]$ so MVT applies:

$\therefore$ since $\dfrac{g(-5)-g(-3)}{5-(-3)} = \dfrac{10-2}{-2} = -4$

we are guaranteed a "c" in $(-5,-3)$ where $g'(c) = -4$.