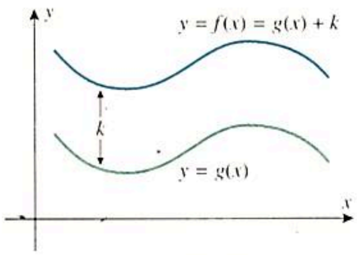


If $f'(x) = g'(x)$ on an interval, then the graphs of f and g are vertical translations of one another.

Figure 6.5.6

Proof. Let $h(x) = f(x) - g(x)$. Then for every x in (a, b)

$$h'(x) = f'(x) - g'(x) = 0$$

Thus, $h(x) = f(x) - g(x)$ is constant on $[a, b]$ by Theorem 5.1.2(c). ∎

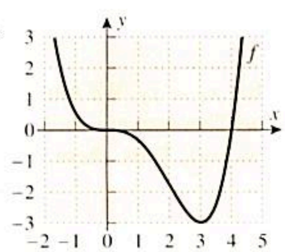
REMARK. This theorem remains true if the closed interval $[a, b]$ is replaced by a finite or infinite interval (a, b), $[a, b)$, or $(a, b]$, provided f and g are differentiable on (a, b) and continuous on the entire interval.

The Constant Difference Theorem has a simple geometric interpretation—it tells us that if f and g have the same derivative on an interval, then there is a constant k such that $f(x) = g(x) + k$ for each x in the interval; that is, the graphs of f and g can be obtained from one another by a vertical translation (Figure 6.5.6).

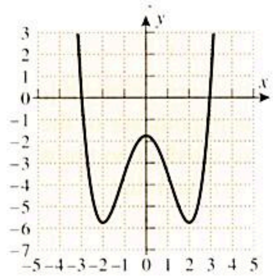
EXERCISE SET 6.5 Graphing Calculator C CAS

In Exercises 1 and 2, use the graph of f to find an interval $[a, b]$ on which Rolle's Theorem applies, and find all values of c in that interval that satisfy the conclusion of the theorem.

1.



2.



In Exercises 3–8, verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

3. $f(x) = x^2 - 6x + 8$; $[2, 4]$

4. $f(x) = x^3 - 3x^2 + 2x$; $[0, 2]$

5. $f(x) = \cos x$; $[\pi/2, 3\pi/2]$

6. $f(x) = \dfrac{x^2 - 1}{x - 2}$; $[-1, 1]$

7. $f(x) = \frac{1}{2}x - \sqrt{x}$; $[0, 4]$

8. $f(x) = \dfrac{1}{x^2} - \dfrac{4}{3x} + \dfrac{1}{3}$; $[1, 3]$

9. Use the graph of f in the accompanying figure to estimate all values of c that satisfy the conclusion of the Mean-Value Theorem on the interval $[0, 8]$.

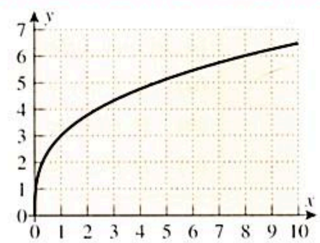


Figure Ex-9

10. Use the graph of f in Exercise 9 to estimate all values of c that satisfy the conclusion of the Mean-Value Theorem on the interval $[0, 4]$.

In Exercises 11–16, verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

11. $f(x) = x^2 + x$; $[-4, 6]$

12. $f(x) = x^3 + x - 4$; $[-1, 2]$

13. $f(x) = \sqrt{x + 1}$; $[0, 3]$

14. $f(x) = x + \dfrac{1}{x}$; $[3, 4]$

15. $f(x) = \sqrt{25 - x^2}$; $[-5, 3]$

16. $f(x) = \dfrac{1}{x - 1}$; $[2, 5]$

17. (a) Find an interval $[a, b]$ on which

$$f(x) = x^4 + x^3 - x^2 + x - 2$$

satisfies the hypotheses of Rolle's Theorem.

(b) Generate the graph of $f'(x)$, and use it to make rough estimates of all values of c in the interval obtained in part (a) that satisfy the conclusion of Rolle's Theorem.

(c) Use Newton's Method to improve on the rough estimates obtained in part (b).

18. Let $f(x) = x^3 + 4x$.

(a) Find the equation of the secant line through the points $(-2, f(-2))$ and $(1, f(1))$.

(b) Show that there is only one number c in the interval $(-2, 1)$ that satisfies the conclusion of the Mean-Value Theorem for the secant line in part (a).

(c) Find the equation of the tangent line to the graph of f at the point $(c, f(c))$.

(d) Use a graphing utility to generate the secant line in part (a) and the tangent line in part (c) in the same coordinate system, and confirm visually that the two lines seem parallel.

19. Let $f(x) = \tan x$.

(a) Show that there is no point c in the interval $(0, \pi)$ such that $f'(c) = 0$, even though $f(0) = f(\pi) = 0$.

(b) Explain why the result in part (a) does not violate Rolle's Theorem.

20. Let $f(x) = x^{2/3}$, $a = -1$, and $b = 8$.

(a) Show that there is no point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) Explain why the result in part (a) does not violate the Mean-Value Theorem.

21. (a) Show that if f is differentiable on $(-\infty, +\infty)$, and if $y = f(x)$ and $y = f'(x)$ are graphed in the same coordinate system, then between any two x-intercepts of f there is at least one x-intercept of f'.

(b) Give some examples that illustrate this.

22. Review Definitions 3.1.3 and 3.1.4 of average and instantaneous rate of change of y with respect to x, and use the Mean-Value Theorem to show that if f is differentiable on $(-\infty, +\infty)$, then in any interval $[x_0, x_1]$ there is at least one point where the instantaneous rate of change of y with respect to x is equal to the average rate of change over the interval.

In Exercises 23–25, use the result of Exercise 22.

23. An automobile travels 4 mi along a straight road in 5 min. Show that the speedometer read exactly 48 mi/h at least once during the trip.

24. At 11 A.M. on a certain morning the outside temperature was 76° F. At 11 P.M. that evening it had dropped to 52° F.

(a) Show that at some instant during this period the temperature was decreasing at the rate of 2° F/h.

(b) Suppose that you know that the temperature reached a high of 88° F sometime between 11 A.M. and 11 P.M. Show that at some instant during this period the temperature was decreasing at a rate greater than 3° F/h.

25. Suppose that two runners in a 100-m dash finish in a tie. Show that they had the same velocity at least once during the race.

26. Use the fact that

$$\frac{d}{dx}(x^6 - 2x^2 + x) = 6x^5 - 4x + 1$$

to show that the equation $6x^5 - 4x + 1 = 0$ has at least one solution in the interval $(0, 1)$.

27. (a) Use the Constant Difference Theorem (6.5.3) to show that if $f'(x) = g'(x)$ for all x in the interval $(-\infty, +\infty)$, and if f and g have the same value at any point x_0, then $f(x) = g(x)$ for all x in $(-\infty, +\infty)$.

(b) Use the result in part (a) to prove the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

28. (a) Use the Constant Difference Theorem (6.5.3) to show that if $f'(x) = g'(x)$ for all x in $(-\infty, +\infty)$, and if $f(x_0) - g(x_0) = c$ at some point x_0, then

$$f(x) - g(x) = c$$

for all x in $(-\infty, +\infty)$.

(b) Use the result in part (a) to show that the function

$$h(x) = (x - 1)^3 - (x^2 + 3)(x - 3)$$

is constant for all x in $(-\infty, +\infty)$, and find the constant.

(c) Check the result in part (b) by multiplying out and simplifying the formula for $h(x)$.

29. (a) Use the Mean-Value Theorem to show that if f is differentiable on an interval I, and if $|f'(x)| \le M$ for all values of x in I, then

$$|f(x) - f(y)| \le M|x - y|$$

for all values of x and y in I.

(b) Use the result in part (a) to show that

$$|\sin x - \sin y| \le |x - y|$$

for all real values of x and y.

30. (a) Use the Mean-Value Theorem to show that if f is differentiable on an open interval I, and if $|f'(x)| \ge M$ for all values of x in I, then

$$|f(x) - f(y)| \ge M|x - y|$$

for all values of x and y in I.

(b) Use the result in part (a) to show that

$$|\tan x - \tan y| \ge |x - y|$$

for all values of x and y in the interval $(-\pi/2, \pi/2)$.

(c) Use the result in part (b) to show that

$$|\tan x + \tan y| \ge |x + y|$$

for all values of x and y in the interval $(-\pi/2, \pi/2)$.