Name: ______________________________ Date:__________

AP Calculus AB Ms. Loughran

1. Let f be the function defined by $f(x) = xe^{1-x}$ for all real numbers x.

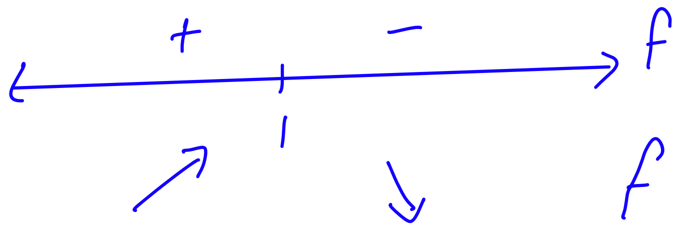
   (a) Find each interval on which f is increasing.
   (b) Find the x-coordinate of each point of inflection of f.
   (c) Using the results found in parts (a) and (b), sketch the graph of f in the xy – plane.

a) $f'(x) = -xe^{1-x} + e^{1-x}$

$0 = -xe^{1-x} + e^{1-x}$

$0 = e^{1-x}(-x+1)$

$e^{1-x} \neq 0$ | $-x+1$, $x = 1$



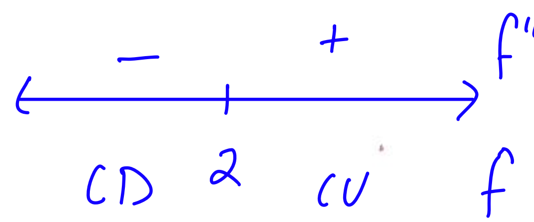
$(-\infty, 1)$

b) $f''(x) = e^{1-x}(-1) - e^{1-x}(-x+1)$

$f''(x) = -e^{1-x} - e^{1-x}(-x+1)$

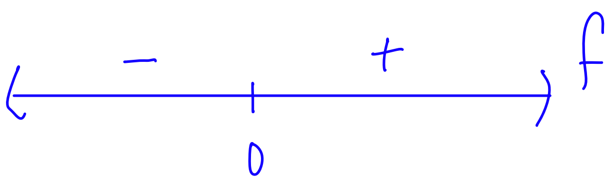
$0 = -e^{1-x}(1 + (-x+1))$ [$-x+2$]

$-e^{1-x} \neq 0$ | $-x+2 = 0$, $x = 2$



$x = 2$

c)

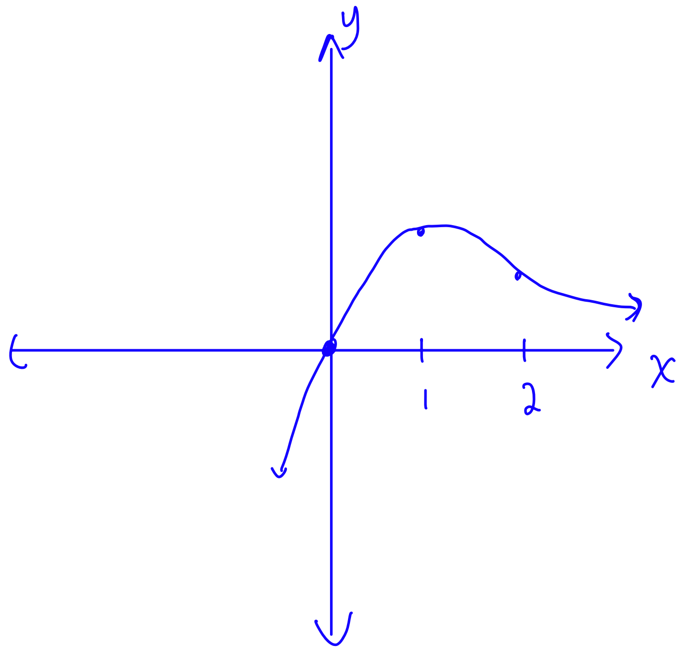


$f(x) = xe^{1-x}$

$0 = xe^{1-x}$

$x = 0$ | $e^{1-x} \neq 0$

$(0,0)$



$f(0) = 0$

$f(1) = 1e^{1-1} = 1$

$f(2) = 2e^{1-2} = \frac{2}{e}$

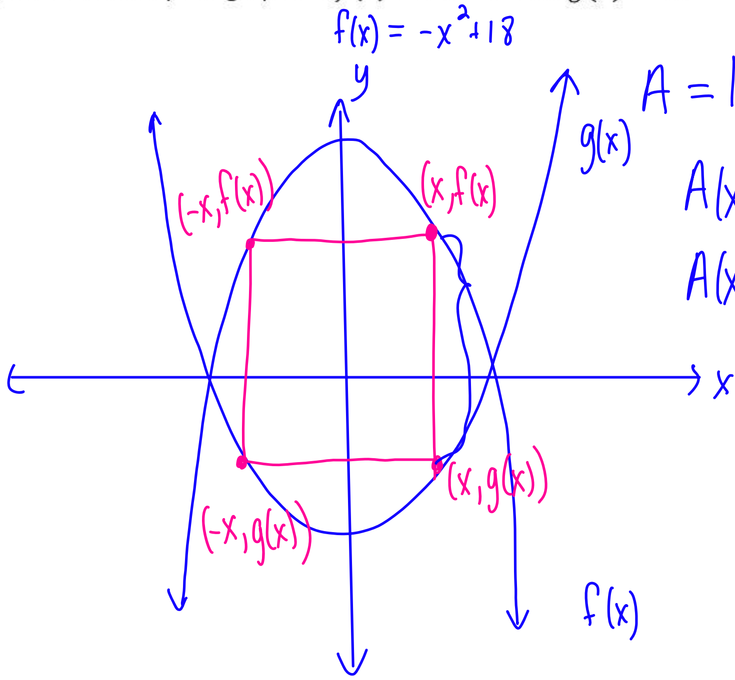
$f(3) = 3e^{1-3} = 3e^{-2} = \frac{3}{e^2}$

Name:______________________________________ Date:_____________

AP Calc AB Min-Max Problems (Sheet 4) Ms. Loughran

1971- AB 4

Find the area of the largest rectangle (with sides parallel to the coordinate axes) that can be inscribed in the region enclosed by the graphs of $f(x) = 18 - x^2$ and $g(x) = 2x^2 - 9$.



$A = lw$

$A(x) = 2x(27 - 3x^2)$

$A(x) = 54x - 6x^3$

$A'(x) = 54 - 18x^2$

$54 - 18x^2 = 0$

$3 - x^2 = 0$

$3 = x^2$

$\pm\sqrt{3} = x$

width $= f(x) - g(x)$

$w = 18 - x^2 - (2x^2 - 9)$

$w = 27 - 3x^2$

$A''(x) = -36x$

$A''(\sqrt{3}) < 0$ ∴ there is a max at $\sqrt{3}$

Restrictions: $(0, 3)$

$2x > 0$

$x > 0$

$27 - 3x^2 > 0$

$9 - x^2 > 0$

$-3 \quad 3$

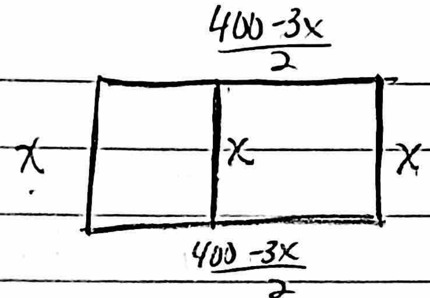
$- < x < 3$

$2\sqrt{3}$ by $27 - 3(\sqrt{3})^2$

Key to More Optimization

$2 \cdot \frac{400-3x}{4} = \frac{400-3x}{2}$ ①

①



$\frac{400-3x}{2}$

$A = lw$

$A(x) = (200 - \frac{3}{2}x)x$

$A(x) = 200x - \frac{3}{2}x^2$

$x > 0$

$A'(x) = 200 - 3x$

$\frac{400-3x}{2} > 0$

$200 - 3x = 0$

$200 - \frac{3}{2}x > 0$

$200 = 3x$

$-\frac{3}{2}x > -200$

$x = 200/3$

$\frac{3}{2}x < 200$

$A''(x) = -3$

$3x < 400$

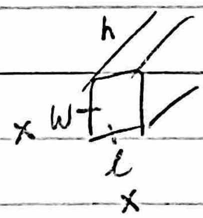
$A''(200/3) = -3 < 0$ ∴ at $x = \frac{200}{3}$ there is a max

$x < 400/3$

$\frac{200}{3}$ ft by $\frac{400 - 3(\frac{200}{3})}{2} = \frac{200}{2} = 100$

$\frac{200}{3}$ ft by 100 ft or $\frac{200}{3}$ ft by 50 ft per corral

②



bottom is a square
open top

$V = lwh$

$x > 0$

$500 = x \cdot x \cdot h$

$\frac{500}{x^2} = h$

$SA = lw + 2lh + 2wh$

$SA(x) = x^2 + 2x \cdot \frac{500}{x^2} + 2x \cdot \frac{500}{x^2}$

$2x - \frac{2000}{x^2} = 0$

$SA(x) = x^2 + 2000x^{-1}$

10ft by 10ft by 5ft

$SA'(x) = 2x - 2000x^{-2}$

$2x = \frac{2000}{x^2}$

$x^3 = 1000$

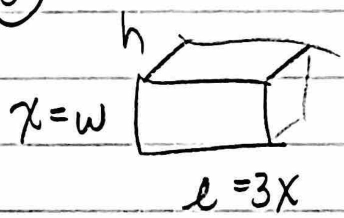
$x = 10$

$2x^3 = 2000$

$SA''(x) = 2 + 4000x^{-3}$ ∴ there is a min at $x = 10$

$SA''(10) > 0$

(3)



$V = lwh$

$50 = x \cdot 3x \cdot h$

$50 = 3x^2 h$

$\frac{50}{3x^2} = h$

$x > 0$

bases ↓

$SA = 2lw + 2lh + 2wh$

$Cost(x) = 10 \cdot 2 \cdot 3x \cdot x + 6 \cdot 2 \cdot 3x \cdot \frac{50}{3x^2} + 6 \cdot 2 \cdot x \cdot \frac{50}{3x^2}$

$Cost(x) = 60x^2 + 600x^{-1} + 200x^{-1}$

$Cost(x) = 60x^2 + 800x^{-1}$

$Cost'(x) = 120x - 800x^{-2}$

$120x - \frac{800}{x^2} = 0$

$120x = \frac{800}{x^2}$

$120x^3 = 800$

$x^3 = \frac{800}{120}$

$x^3 = 20/3$

$x = \sqrt[3]{20/3}$

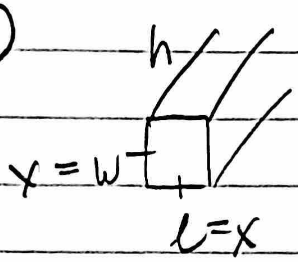
$Cost''(x) = 120 + 1600x^{-3}$

$Cost''(\sqrt[3]{20/3}) > 0$ ∪

∴ there is a min at $\sqrt[3]{\frac{20}{3}}$ ft

$\sqrt[3]{20/3}$ ft by $3\sqrt[3]{20/3}$ ft by $\frac{50}{3\left(\sqrt[3]{\frac{20}{3}}\right)^2}$

(4) *open top*



$SA = lw + 2lh + 2wh$

$SA(x) = x^2 + 2xh + 2xh$

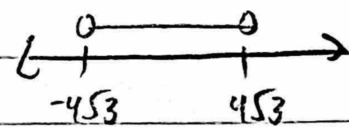
$SA(x) = x^2 + 4xh$

$48 = x^2 + 4xh$

$\frac{48 - x^2}{4x} = h$

$x > 0$

$48 - x^2 > 0$



$0 < x < 4\sqrt{3}$

$V(x) = x \cdot \cancel{x} \cdot \frac{48 - x^2}{4\cancel{x}} = \frac{48x - x^3}{4} = 12x - \frac{1}{4}x^3$

$V'(x) = 12 - \frac{3}{4}x^2$

$12 - \frac{3}{4}x^2 = 0$

$12 = \frac{3}{4}x^2$

$48 = 3x^2$

$x^2 = 16$

$x = \pm 4$ reject -4 outside of interval

$V''(x) = -\frac{6}{4}x$

$V''(4) = \frac{-6}{4} \cdot 4 = -6 < 0$

$\therefore$ at $x = 4$ there is a rel max

4 ft by 4 ft by $\frac{48 - 4^2}{4(4)}$ (16)

$\frac{32}{16}$

4 ft by 4 ft by 2 ft

(5)

$SA = \pi r^2 + 2\pi r h$

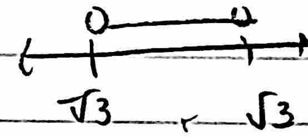
$3\pi = \pi r^2 + 2\pi r h$

$3 = r^2 + 2rh$

$\frac{3-r^2}{2r} = \frac{2rh}{2r}$

$r > 0$

$3 - r^2 > 0$



$V = \pi r^2 h$

$\therefore 0 < r < \sqrt{3}$

$V(r) = \pi r^2 \cdot \frac{3-r^2}{2r} = \frac{\pi}{2}(3r - r^3)$

$V'(r) = \frac{\pi}{2}(3 - 3r^2)$

$V''(r) = \frac{\pi}{2}(-6r)$

$\frac{\pi}{2}(3 - 3r^2) = 0$

$V''(1) < 0$ ∩ ∴ there is a max at $r = 1$

$3 - 3r^2 = 0$

$3 = 3r^2$

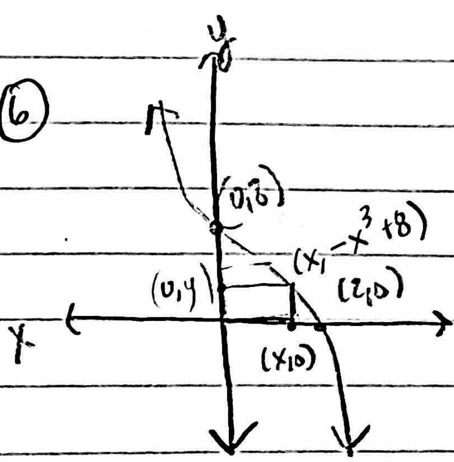
$1 = r^2$

$r = \pm 1$ reject -1

height 1 ft

radius = 1 ft

(6)



$y = 8 - x^3$

$y = -x^3 + 8$

x^3 reflected over x-axis ↑ 8

x-intercept : $0 = -x^3 + 8$

$x^3 = 8$

$x = 2$

$A(x) = x(-x^3 + 8)$

$0 < x < 2$

$A(x) = -x^4 + 8x$

$A'(x) = -4x^3 + 8$

↓

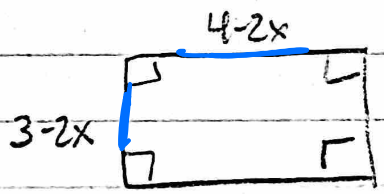
$$-4x^3+8=0$$
$$-4x^3=-8$$
$$x^3=2$$
$$x=\sqrt[3]{2}$$

$$A''(x)=-12x^2$$
$A''(\sqrt[3]{2})<0$ $\cap$ $\therefore$ there is a max at $x=\sqrt[3]{2}$

$\sqrt[3]{2}$ by $8-(\sqrt[3]{2})^3$

$\sqrt[3]{2}$ by 6

7



$$V(x)=x(3-2x)(4-2x)$$
$$V(x)=(3x-2x^2)(4-2x)$$
$$V(x)=12x-6x^2-8x^2+4x^3$$
$$V(x)=4x^3-14x^2+12x$$
$$V'(x)=12x^2-28x+12$$
$$12x^2-28x+12=0$$
$$4(3x^2-7x+3)=0$$
$$x=\frac{7\pm\sqrt{(7)^2-4(3)(3)}}{2(3)}=\frac{7\pm\sqrt{13}}{6}$$

reject $\frac{7+\sqrt{13}}{6}$

$x>0$

$3-2x>0$ $\quad$ $4-2x>0$

$-2x>-3$ $\quad$ $-2x>-4$

$x<3/2$ $\quad$ $x<2$

$0<x<3/2$

$$V''(x)=24x-28$$
$$V''\left(\frac{7-\sqrt{13}}{6}\right)=24\left(\frac{7-\sqrt{13}}{6}\right)-28=28-4\sqrt{13}-28<0\ \cap$$

$\therefore$ there is a max at $\frac{7-\sqrt{13}}{6}$

$\left(3-2\left(\frac{7-\sqrt{13}}{6}\right)\right)$ ft by $\left(4-2\left(\frac{7-\sqrt{13}}{6}\right)\right)$ ft by $\frac{7-\sqrt{13}}{6}$ ft

Name: Key

AP Calculus Build a Snowman

Date: ____________

Ms. Loughran

x	0	1	2	3
$f''(x)$	5	0	−7	4

1. The polynomial function f has selected values of its second derivative given in the table above. Which of the following statements must be true?

   (A) The graph of f changes concavity in the interval (0, 2).
   (B) f is decreasing on the interval (0, 2).
   (C) f has a local minimum at $x = 1$.
   (D) The graph of f has a point of inflection at $x = 1$.

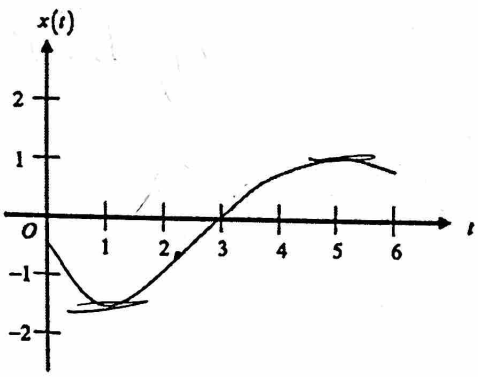
2. If $f(x) = (x-1)(x^2+2)^3$, then $f'(x) =$ $3(x-1)(x^2+2)^2 \cdot 2x + (x^2+2)^3$

   $= 6x(x-1)(x^2+2)^2 + (x^2+2)^3$

   $(x^2+2)^2[6x^2-6x+x^2+2]$

   $(x^2+2)^2[7x^2-6x+2)$

   (A) $6x(x^2+2)^2$
   (B) $(x^2+2)^2(7x^2-6x+2)$
   (C) $(x^2+2)^2(x^2+3x-1)$
   (D) $6x(x-1)(x^2+2)^2$



3. A particle moves along a straight line. The graph of the particle's position $x(t)$ at time t is shown above for $0 < t < 6$. The graph has horizontal tangents at $t = 1$ and $t = 5$ and a point of inflection at $t = 2$. For what values of t is the velocity of the particle increasing?

   $v(t) \nearrow$ where $v''(t)$ +

   where $x(t)$ is concave up

   (A) $0 < t < 2$ (B) $1 < t < 5$ (C) $2 < t < 6$ (D) $3 < t < 5$

$$f(x)=\begin{cases} cx+d & \text{for } x\le 2 \\ x^2-cx & \text{for } x>2 \end{cases}$$

$2c+d = 4-2c$
$4+d=0$
$d=-4$

$f'(x)=\begin{cases} c \\ 2x-c \end{cases}$

4. Let f be the function defined above, where c and d are constants. If f is differentiable at $x = 2$, what is the value of $c+d$?

(A) −4　　(B) −2　　(C) 0　　(D) 2

$c=2(2)-c$
$2c=4$
$c=2$

$2+(-4)=-2$

5. What are all horizontal asymptotes of the graph of $y=\dfrac{5+2^x}{1-2^x}$ in the xy-plane?

(A) $y=-1$ only
(B) $y=5$ only
(C) $y=-1$ and $y=5$
(D) $y=-1$ and $y=0$

$\lim_{x\to-\infty} \dfrac{5+2^x}{1-2^x} = \dfrac{5+\frac{1}{2^x}}{1-\frac{1}{2^x}} = 5$　　$\lim_{x\to\infty} \dfrac{2^x \ln 2}{-2^x \ln 2} = -1$

6. (CA) The first derivative of the function f is defined by $f'(x)=\sin(x^3-x)$ for $0\le x\le 2$. On what interval(s) is f increasing?

(A) $1\le x\le 1.445$
(B) $1\le x\le 1.691$
(C) $0.577\le x\le 1.445$ and $1.875\le x\le 2$
(D) $0\le x\le 1$ and $1.691\le x\le 2$

$(1,\ 1.6906\ldots$

7. What is the slope of the line tangent to the cure $y=\arctan(4x)$ at the point $x=\dfrac{1}{4}$?

(A) 2　　(B) -2　　(C) $\frac{1}{2}$　　(D) $-\frac{1}{2}$

$y'=\dfrac{1}{(4x)^2+1}\cdot 4 = \dfrac{4}{16x^2+1}$

$y'\left(\frac{1}{4}\right) = \dfrac{4}{2} = 2$

8. Let f be a function with a second derivative given by $f''(x) = x^2(x-3)(x-6)$. What are the x-coordinates of the points of inflection of the graph of f?

(A) 0 only
(B) 3 only
(C) 0, 3, and 6
(D) 3 and 6 only

$$f(x) = \begin{cases} cx + d & \text{for } x \le 2 \\ x^2 - cx & \text{for } x > 2 \end{cases}$$

9. Let f be the function defined above. Which of the following statements about f are true?

I. f has a limit at $x = 2$.

II. f is continuous at $x = 2$.

III. f is differentiable at $x = 2$.

(A) I only
(B) II only
(C) III only
(D) I and II only

10. In the xy-plane, the line $x + y = k$, where k is a constant, is tangent to the graph of $y = x^2 + 3x + 1$. What is the value of k?

(A) −3 (B) −2 (C) −1 (D) 0

11. (CA) The derivative of the function f is given by $f'(x) = x^2\cos(x^2)$. How many points of inflection does the graph of f have on the open interval $(-2, 2)$?

(A) Two (B) Three (C) Five (D) Four

$y-1=4(x-2)$

$y-1=4(1.9-2)$

12. The function f is twice differentiable with $f(2)=1$, $f'(2)=4$ and $f''(2)=3$. What is the value of the approximation of $f(1.9)$ using the tangent to the graph of f at $x=2$?

(A) 0.4 (B) 0.6 (C) 0.7 (D) 1.3

$\cos(xy)\cdot(xy'+y)=1$

13. If $\sin(xy)=x$, then $\frac{dy}{dx}=$

(A) $\frac{1}{\cos(xy)}$ (B) $\frac{1}{x\cos(xy)}$ (C) $\frac{1-\cos(xy)}{\cos(xy)}$ (D) $\frac{1-y\cos(xy)}{x\cos(xy)}$

$xy'+y=\frac{1}{\cos(xy)}$

$\frac{dr}{dt}=-2\text{cm/s}$

14. **(CA)** The radius of a sphere is decreasing at a rate of 2 centimeters per second. At the instant when the radius of the sphere is 3 centimeters, what is the rate of change, in square centimeters per second, of the surface area of the sphere?

(A) -108π (B) -72π (C) -48π (D) -24π

15. Let f be a differentiable function such that $f(3)=15$, $f(6)=3$, $f'(3)=-8$, and $f'(6)=-2$. The function g is differentiable and $g(x)=f^{-1}(x)$ for all x. What is the value of $g'(3)$?

(A) $-\frac{1}{2}$ (B) $-\frac{1}{8}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$

$g'(3)=\frac{1}{f'(6)}$

$=\frac{1}{-2}$

(13) $\cos(xy)\left(x\frac{dy}{dx}+y\right)=1$

$x\frac{dy}{dx}+y=\frac{1}{\cos(xy)}$

$\frac{x\frac{dy}{dx}}{x}=\frac{\frac{1}{\cos(xy)}-y}{x}$

$\frac{1-y\cos(xy)}{x\cos(xy)}$