Do Now:

LINEAR MOTION REVIEW

2. The position (x-coordinate) of a particle moving on the line $y = 2$ is given by $x(t) = t^3 - 3t^2 - 9t + 2$ where t is the time in seconds, $t \geq 0$ and x is the position in feet from the point (0,2).

(a) When does the particle change direction?
(b) What is the total distance traveled by the particle on the interval [0,4]?
(c) What is the particle's acceleration at $t = 2$?
(d) On [-1,3], when is the SPEED of the particle a maximum?

(a) $v(t) = 3t^2 - 6t - 9$

$3t^2 - 6t - 9 = 0$

$t^2 - 2t - 3 = 0$

$(t+1)(t-3) = 0$

$t = \cancel{-1}, 3$

$v(t)$	0	$-$	3	$+$
		left		right

$t = 3$ sec.

(c) $a(t) = 6t - 6$

$a(2) = 6 \text{ ft/s}^2$

(d) speed = |velocity|

we need to find the maximum & minimum velocity

$v'(t) = a(t) = 6t - 6$

$6t - 6 = 0$

$t = 1$

$t \neq -1$

$v(-1)$

$v(1) = -12$

$v(3) = 27 - 18 - 9 = 0$

maximum speed = 12 ft/s

Mean Value Theorem: MVT

If $y = f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable on the open interval (a,b) then there is at least one point c in (a,b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

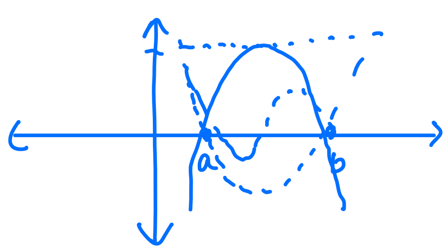
instaneous rate of change

average rate of change

Rolle's Theorem: (a special case of MVT)

Let f be diff on (a,b) and continuous on $[a,b]$.

If $f(a) = f(b) = 0$ then there is at least one point c in (a,b) where $f'(c) = 0$



Name: ______________________________ Date: ________

Calculus AB Mean-Value Theorem Ms. Loughran

Check that the hypotheses of the Mean-Value Theorem are satisfied on the given interval. If so, find all the values of c in that interval that satisfy the conclusion of the theorem.

1. $f(x)=x^2+x$; $[-4,6]$

continuous $[-4,6]$ ✓
diff $(-4,-6)$ ✓

$f'(x)=2x+1$

$2c+1=\frac{f(6)-f(-4)}{6-(-4)}$

$2c+1=\frac{42-12}{10}$

$2c+1=3$
$2c=2$
$c=1$

only an issue if $x<-1$

2. $f(x)=\sqrt{x+1}$; $[0,3]$

Cont: $[0,3]$ ✓
diff: $(0,3)$ ✓

$f'(x)=\frac{1}{2}(x+1)^{-\frac{1}{2}}=\frac{1}{2\sqrt{x+1}}$

$\frac{1}{2\sqrt{c+1}}=\frac{\overset{2}{f(3)}-\overset{1}{f(0)}}{3}$

$\frac{1}{2\sqrt{c+1}}=\frac{1}{3}$

$(2\sqrt{c+1})^2=(3)^2$
$4(c+1)=9$
$4c+4=9$
$4c=5$
$c=\frac{5}{4}$

$y=\sqrt{25-x^2}$
$y^2=25-x^2$
$x^2+y^2=25$

3. $f(x)=\sqrt{25-x^2}$; $[-5,3]$

continuous: $[-5,3]$ ✓
diff. $(-5,3)$ ✓

$f'(x)=\frac{1}{\cancel{2}}(25-x^2)^{-\frac{1}{2}}\cdot-\cancel{2}x$

$f'(x)=\frac{-x}{\sqrt{25-x^2}}$

-5 is not in the interval

$\frac{-c}{\sqrt{25-c^2}}=\frac{f(3)-f(-5)}{3+5}$

$\frac{-c}{\sqrt{25-c^2}}=\frac{4-0}{8}$

$\frac{-c}{\sqrt{25-c^2}}=\frac{1}{2}$

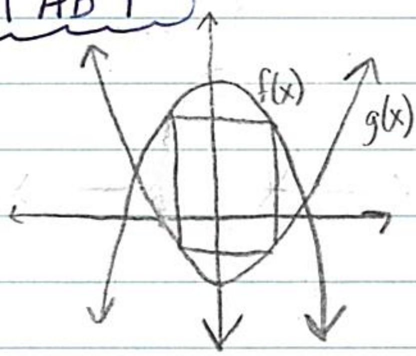
c has to be ⊖

$-2c=\sqrt{25-c^2}$
$4c^2=25-c^2$
$5c^2=25$
$c^2=5$
$c=\cancel{\pm}\sqrt{5}$

4. $f(x)=2x^{\frac{1}{4}}$; $[-2,1]$

Optimization Sheet 4 Key

1971 AB 4



$A(x) = 2x\,[f(x) - g(x)]$

$A(x) = 2x\,(18 - x^2 - (2x^2 - 9))$

$A(x) = 2x\,(27 - 3x^2)$

$A(x) = 54x - 6x^3$

Restrictions:

$x > 0$

$f(x) = g(x)$

$18 - x^2 = 2x^2 - 9$

$27 = 3x^2$

$9 = x^2$

$\pm 3 = x$

$0 < x < 3$

$A'(x) = 54 - 18x^2$

$54 - 18x^2 = 0$

$54 = 18x^2$

$3 = x^2$

$\pm\sqrt{3} = x$

$A''(x) = -36x$

$A''(\sqrt{3}) < 0$ ∩

$A'(x)$: $-$ on $(-\infty, -\sqrt{3})$, $+$ on $(-\sqrt{3}, \sqrt{3})$, $-$ on $(\sqrt{3}, \infty)$

$-\sqrt{3}$: ↘ ↗ rel min

$\sqrt{3}$: ↗ ↘ rel max

$A(\sqrt{3}) = 2\sqrt{3}\,(27 - 3(\sqrt{3})^2)$

$A(\sqrt{3}) = 2\sqrt{3}\,(27 - 9)$

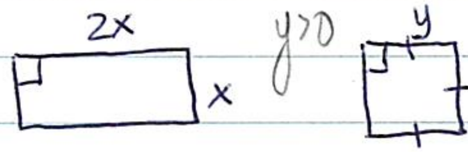
$A(\sqrt{3}) = 36\sqrt{3}$ units^2

1972 AB 4

$x>0$

$y>0$

(a)



(a) $20 \le x \le 50$

$2x(x) \ge 800$

$2x^2 \ge 800$

$x^2 \ge 400$

$x \ge 20$

$y(y) \ge 100$

$y^2 \ge 100$

$y \ge 10$

$P = 45$ if $y \ge 10$

y is at least 10

$340 - 40 = 300$

$300 = 6x$

$x = 50$

$\dfrac{170-3x}{2} \ge 10$

$170 - 3x \ge 20$

$-3x \ge -150$

$x \le 50$

(b) $2(2x) + 2x + 4y = 340$

$6x + 4y = 340$

$4y = 340 - 6x$

$y = \dfrac{340-6x}{4} = \dfrac{170-3x}{2}$

$A(x) = 2x^2 + \left(\dfrac{170-3x}{2}\right)^2$

$A(x) = 2x^2 + (85 - 1.5x)^2$

$A(x) = 2x^2 + 7225 - 255x + 2.25x^2$

$A(x) = 4.25x^2 - 255x + 7225$

$A'(x) = 8.5x - 255$

$0 = 8.5x - 255$

$255 = 8.5x$

$30 = x$

$A'(x)$: $-$ before 30, $+$ after 30

$A(20) = 3825 \text{ yds}^2$

$A(30) = 3400 \text{ yds}^2$

$A(50) = 5100 \text{ yds}^2$

$x=30$ is a rel. min. So you know that max has to be at an endpoint.

1973 AB 6

Profit = Price - cost

$$P(x) = \begin{cases} 13x - (x^2+5x+7) = -x^2+8x-7, & 0 \le x \le 3 \\ 13x - (x^2+5x+7+3(x-3)) = -x^2+5x+2, & 3 < x \le 10 \end{cases}$$

$$P'(x) = \begin{cases} -2x+8, & 0 \le x \le 3 \\ -2x+5 & 3 < x \le 10 \end{cases}$$

$-2x+8=0$
$2x=8$
$x=4$
$\therefore$ outside its domain

$-2x+5=0$
$2x=5$
$x=5/2$
outside of its domain

$\lim_{x\to 3^+} P'(x) = -2(3)+5 = -1$

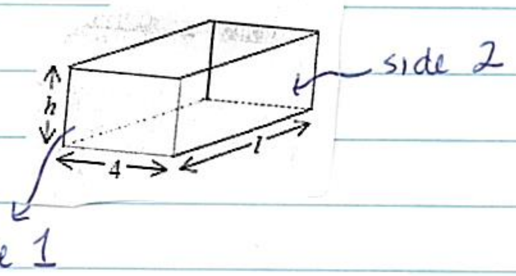
$\lim_{x\to 3^-} P'(x) = -2(3)+8 = 2$

$\therefore$ $P'(3)$ does not exist
and $x=3$ is then a critical value

Candidate test
$P(0) = -(0)^2+8(0)-7 = -7$
$P(3) \quad -(3)^2+8(3)-7 = -9+24-7 = 8$
$P(10) = -(10)^2+5(10)+2 = -48$

$\therefore$ A 3 ton level of output maximizes profit.

1982 AB 6



Cost of base: $10wl = 10 \cdot 4 \cdot \frac{9}{h} = \frac{360}{h}$

Cost of side 1: $5wh = 5 \cdot 4 \cdot h = 20h$

Cost of side 2: $5lh = 5(9) = 45$

* only one base - open top box

$V = lwh$

$36 = 4lh$

$\frac{36}{4} = lh$

$9 = lh$

$l = \frac{9}{h}$

$C(h) = 360h^{-1} + 2(20h + 45)$

$C(h) = 360h^{-1} + 40h + 90$

$C'(h) = -360h^{-2} + 40$

$\frac{-360}{h^2} + 40 = 0$

$\frac{-360}{h^2} = -40$

$-40h^2 = -360$

$h^2 = 9$

$h = \pm 3$ reject -3

$-$ $+$ C'

3

$\therefore$ at $h = 3$ there is a minimum

$C(3) = \frac{360}{3} + 40(3) + 90 = \$330$

Homework 01-05

1997 AP Calculus AB: Section I, Part A

50 Minutes—No Calculator

Note: Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

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2. If $f(x) = x\sqrt{2x-3}$, then $f'(x) =$ $\sqrt{2x-3} + \frac{1}{2}x(2x-3)^{-\frac{1}{2}}(2)$

$= \sqrt{2x-3} + \dfrac{2x}{2\sqrt{2x-3}}$

$\dfrac{2(2x-3)+2x}{2\sqrt{2x-3}}$

$\dfrac{4x-6+2x}{2\sqrt{2x-3}}$

$\dfrac{6x-6}{2\sqrt{2x-3}} = \dfrac{\cancel{2}(3x-3)}{\cancel{2}\sqrt{2x-3}}$

(A) $\dfrac{3x-3}{\sqrt{2x-3}}$

(B) $\dfrac{x}{\sqrt{2x-3}}$

(C) $\dfrac{1}{\sqrt{2x-3}}$

(D) $\dfrac{-x+3}{\sqrt{2x-3}}$

(E) $\dfrac{5x-6}{2\sqrt{2x-3}}$

---

4. If $f(x) = -x^3 + x + \frac{1}{x}$, then $f'(-1) =$ $-3(-1)^2+1-\frac{1}{1^2}$

$f'(x) = -3x^2+1-\frac{1}{x^2}$

$-3\cancel{+1}\cancel{-1}$

(A) 3 (B) 1 (C) −1 (D) −3 (E) −5

5. The graph of $y = 3x^4 - 16x^3 + 24x^2 + 48$ is concave down for

(A) $x < 0$

(B) $x > 0$

(C) $x < -2$ or $x > -\frac{2}{3}$

(D) $x < \frac{2}{3}$ or $x > 2$

(E) $\frac{2}{3} < x < 2$

$y' = 12x^3 - 48x^2 + 48x$

$y'' = 36x^2 - 96x + 48$

$36x^2 - 96x + 48 = 0$

$12(3x^2 - 8x + 4) = 0$

$12(3x-2)(x-2) = 0$

$x = \frac{2}{3}$ $\quad x = 2$

$+ \quad - \quad + \quad y''$

$\frac{2}{3} \quad 2$

CU CD CU y

---

7. $\frac{d}{dx}\cos^2(x^3) = 2\cos(x^3)(-\sin(x^3)) \cdot 3x^2$

(A) $6x^2 \sin(x^3)\cos(x^3)$

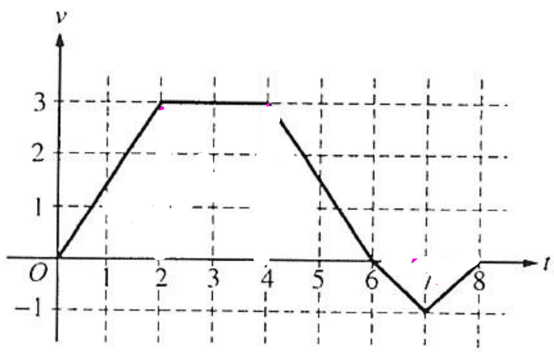
(B) $6x^2 \cos(x^3)$

(C) $\sin^2(x^3)$

(D) $-6x^2 \sin(x^3)\cos(x^3)$

(E) $-2\sin(x^3)\cos(x^3)$

Questions 8-9 refer to the following situation.



A bug begins to crawl up a vertical wire at time $t = 0$. The velocity v of the bug at time t, $0 \le t \le 8$, is given by the function whose graph is shown above.

$v \rightarrow +$ to $-$ or $-$ to $+$

8. At what value of t does the bug change direction?

(A) 2 (B) 4 (C) 6 (D) 7 (E) 8

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10. An equation of the line tangent to the graph of $y = \cos(2x)$ at $x = \frac{\pi}{4}$ is

(A) $y - 1 = -\left(x - \frac{\pi}{4}\right)$

(B) $y - 1 = -2\left(x - \frac{\pi}{4}\right)$

(C) $y = 2\left(x - \frac{\pi}{4}\right)$

(D) $y = -\left(x - \frac{\pi}{4}\right)$

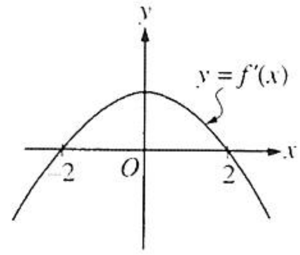
(E) $y = -2\left(x - \frac{\pi}{4}\right)$

$y\left(\frac{\pi}{4}\right) = \cos\left(2 \cdot \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$ $\left(\frac{\pi}{4}, 0\right)$

$y' = -2\sin(2x)$

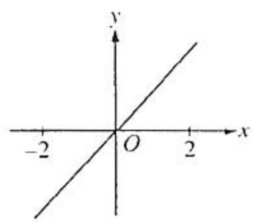
$y'\left(\frac{\pi}{4}\right) = -2\sin\left(\frac{2\pi}{4}\right) = -2\sin\frac{\pi}{2}$
$= -2(1) = -2$

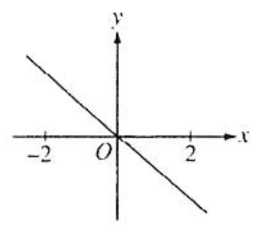
$y - 0 = -2\left(x - \frac{\pi}{4}\right)$

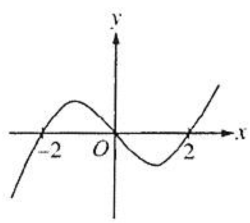


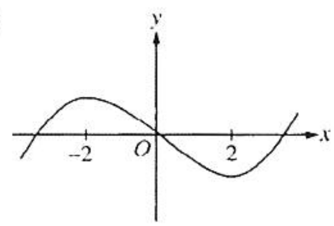
f' − 0 + 0 −
f ↘ HTL ↗ HTL ↘
rel min   rel max

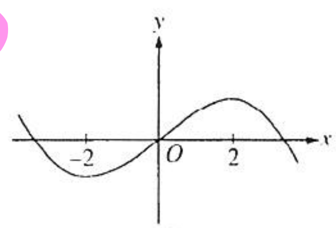
11. The graph of the derivative of f is shown in the figure above. Which of the following could be the graph of f?

(A)


(B)


(C)


(D)


(E)


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12. At what point on the graph of $y=\frac{1}{2}x^2$ is the tangent line parallel to the line $2x-4y=3$?

(A) $\left(\frac{1}{2},-\frac{1}{2}\right)$ (B) $\left(\frac{1}{2},\frac{1}{8}\right)$ (C) $\left(1,-\frac{1}{4}\right)$ (D) $\left(1,\frac{1}{2}\right)$ (E) $(2,2)$

$y' = x$
$x = \frac{1}{2}$
$y = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$

$4y = 2x - 3$
$y = \frac{1}{2}x - \frac{3}{4}$
$m = \frac{1}{2}$

always +

13. Let f be a function defined for all real numbers x. If $f'(x) = \dfrac{\left|4 - x^2\right|}{x - 2}$, then f is decreasing on the interval

− − + f'
−2 2

(A) $(-\infty, 2)$ (B) $(-\infty, \infty)$ (C) $(-2, 4)$ (D) $(-2, \infty)$ (E) $(2, \infty)$

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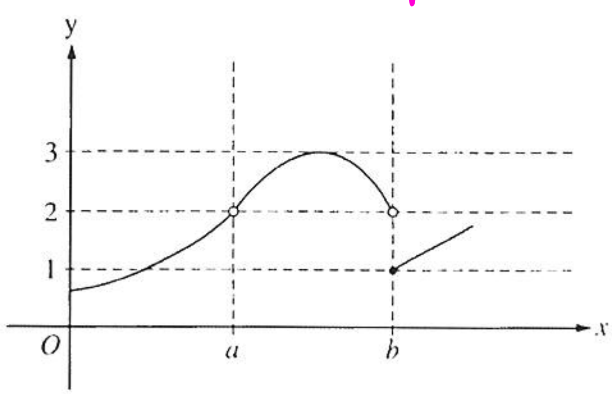
14. Let f be a differentiable function such that $f(3) = 2$ and $f'(3) = 5$. If the tangent line to the graph of f at $x = 3$ is used to find an approximation to a zero of f, that approximation is

$y-2=5(x-3)$

(A) 0.4 (B) 0.5 (C) 2.6 (D) 3.4 (E) 5.5

$y=0$

$y=5(x-3)+2$
$0=5x-15+2$
$13=5x$
$x=\frac{13}{5}=2.6$



15. The graph of the function f is shown in the figure above. Which of the following statements about f is true?

(A) $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to b} f(x)$ x

(B) $\lim\limits_{x \to a} f(x) = 2$

(C) $\lim\limits_{x \to b} f(x) = \cancel{2}$ dne

(D) $\lim\limits_{x \to b} f(x) = \cancel{1}$ dne

(E) $\lim\limits_{x \to a} f(x)$ does ~~not~~ exist. 2

17. If $x^2 + y^2 = 25$, what is the value of $\dfrac{d^2y}{dx^2}$ at the point $(4, 3)$?

(A) $-\dfrac{25}{27}$ (B) $-\dfrac{7}{27}$ (C) $\dfrac{7}{27}$ (D) $\dfrac{3}{4}$ (E) $\dfrac{25}{27}$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

$\left.\frac{dy}{dx}\right|_{(4,3)} = \frac{-4}{3}$

$\frac{d}{dx}\left[\frac{-x}{y}\right]$

$= \frac{y(-1) - (-x)\left(\frac{dy}{dx}\right)}{y^2}$

$\frac{3(-1) - (-4)\left(-\frac{4}{3}\right)}{3^2}$

$\frac{-3 - \frac{16}{3}}{9}$

$\frac{-9 - 16}{27}$

19. If $f(x) = \ln\left|x^2 - 1\right|$, then $f'(x) =$

(A) $\left|\dfrac{2x}{x^2 - 1}\right|$

(B) $\dfrac{2x}{\left|x^2 - 1\right|}$

(C) $\dfrac{2|x|}{x^2 - 1}$

(D) $\dfrac{2x}{x^2 - 1}$

(E) $\dfrac{1}{x^2 - 1}$

$|x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } x^2 - 1 > 0 \\ -(x^2 - 1) & x^2 - 1 < 0 \end{cases}$

$\left[\ln(x^2 - 1)\right]' = \frac{1}{x^2 - 1} \cdot 2x$

$\left[\ln(-x^2 + 1)\right]' = \frac{1}{\cancel{-}(x^2 - 1)} \cdot \cancel{-}2x$

21. $\lim\limits_{x \to 1} \dfrac{x}{\ln x}$ is

$\dfrac{1}{0}$

.9	1	1.1
$-\infty$	VA	$+\infty$

(A) 0 (B) $\frac{1}{e}$ (C) 1 (D) e (E) nonexistent

22. What are all values of x for which the function f defined by $f(x) = (x^2 - 3)e^{-x}$ is increasing?

(A) There are no such values of x.

(B) $x < -1$ and $x > 3$

(C) $-3 < x < 1$

(D) $-1 < x < 3$

(E) All values of x

$f'(x) = -(x^2-3)e^{-x} + e^{-x}(2x)$

$f'(x) = -e^{-x}(x^2-3-2x)$

$f'(x) = -e^{-x}(x^2-2x-3)$

$f'(x) = -e^{-x}(x-3)(x+1)$

$0 = -e^{-x}(x-3)(x+1)$

$-e^{-x} \neq 0$ | $x = 3$ | $x = -1$

f': $-$ | -1 | $+$ | 3 | $-$