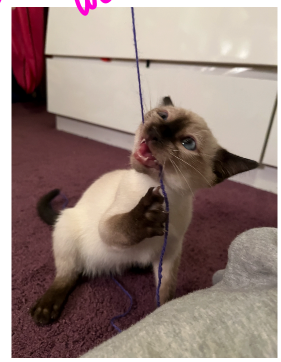
Name: ______________________________ Date: ___________

AP Calculus AB: Integrating Using U-Substitution Ms. Loughran

Welcome Wallie!



Do Now:

1. $\int \frac{dt}{t\sqrt{2t}} =$ $\frac{1}{\sqrt{2}}\int \frac{1}{t\sqrt{t}}\,dt = \frac{1}{\sqrt{2}}\int t^{-3/2}dt = \frac{1}{\sqrt{2}} \cdot -2\,t^{-\frac{1}{2}} + C$

$\frac{-2}{\sqrt{2t}} + C$

Classwork:

1. $\int 2x(x^2+1)^{23}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\int u^{23}\,du = \frac{u^{24}}{24} + C$

$\frac{(x^2+1)^{24}}{24} + C$

2. $\int \frac{3x\,dx}{\sqrt{4x^2+5}}$ $= 3\int \frac{x\,dx}{\sqrt{4x^2+5}}$

$u = 4x^2+5$

$du = 8x\,dx$

$\frac{du}{8} = x\,dx$

$3\int u^{-\frac{1}{2}} \cdot \frac{du}{8} = \frac{3}{8}\int u^{-\frac{1}{2}}du$

$\frac{3}{8}\,2\,u^{\frac{1}{2}} + C$

$\frac{6}{8}\,u^{\frac{1}{2}} + C$

$\frac{6}{8}(4x^2+5)^{\frac{1}{2}} + C$

$\frac{6}{8}\sqrt{4x^2+5} + C$

Steps:

1. look for a piece of the integrand whose derivative is also in the integrand (try denominator, anything being raised to a power, anything under a radical)

2. set u = piece, take the derivative of u with respect to the variable in the expression

3. use u and du expressions to replace the original integral and your new integral should be easier to solve

4. integrate

5. replace u

$\frac{x^2}{x^3-4} \neq \frac{x^2}{x^3} - \frac{x^2}{4}$

3. $\int \frac{x^2}{x^3-4}\,dx$

$u = x^3 - 4$
$du = 3x^2\,dx$
$\frac{du}{3} = x^2\,dx$

$\int \frac{1}{u}\,\frac{du}{3}$

$\frac{1}{3}\int \frac{1}{u}\,du$

$\frac{1}{3}\ln|u| + C$

$\frac{1}{3}\ln|x^3-4| + C$

4. $\int x(2-x^2)^3\,dx$

$u = 2 - x^2$
$du = -2x\,dx$
$\frac{du}{-2} = x\,dx$

$\int u^3\,\frac{du}{-2}$

$-\frac{1}{2}\int u^3\,du = -\frac{1}{2}\cdot\frac{u^4}{4} + C$

$-\frac{(2-x^2)^4}{8} + C$

5. $\int \sin x\, e^{\cos x}\,dx$

$u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

$-\int du\, e^u = -e^u + C$

$-e^{\cos x} + C$

Integration Practice

In Exercises 1 to 15 *use the given substitution* to find the antiderivative.

1. $\int (1+3x)^5 3\,dx$; $u = 1+3x$.

2. $\int \frac{x}{(1+x^2)^3}\,dx$; $u = 1+x^2$.

3. $\int e^{\sin\theta}\cos\theta\,d\theta$; $u = \sin\theta$.

4. $\int \frac{x}{\sqrt{1+x^2}}\,dx$; $u = 1+x^2$.

5. $\int \sqrt{1+x^2}\,x\,dx$; $u = 1+x^2$.

6. $\int \sin 2x\,dx$; $u = 2x$.

7. $\int \frac{e^{2x}}{(1+e^{2x})^2}\,dx$; $u = 1+e^{2x}$.

8. $\int e^{3x}\,dx$; $u = 3x$.

9. $\int \frac{e^{1/x}}{x^2}\,dx$; $u = 1/x$.

11. $\int \frac{t\,dt}{\sqrt{2-5t^2}}$; $u = 2-5t^2$.

12. $\int \tan\theta\sec^2\theta\,d\theta$; $u = \tan\theta$.

13. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$; $u = \sqrt{x}$.

14. $\int \frac{(\ln x)^4}{x}\,dx$; $u = \ln x$.

15. $\int \frac{\sin(\ln x)}{x}\,dx$; $u = \ln x$.

In Exercises 16 to 30 choose an appropriate substitution and find the antiderivative.

16. $\int (1-x^2)^5 x\,dx$

17. $\int \frac{x\,dx}{(x^2+1)^3}$

18. $\int \sqrt[3]{1+x^2}\,x\,dx$

19. $\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$

20. $\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

21. $\int \sin 3\theta\,d\theta$

22. $\int e^x \sin e^x\,dx$

23. $\int \frac{dx}{\sqrt{2x+5}}$

24. $\int (x-3)^{5/2}\,dx$

25. $\int \frac{dx}{(4x+3)^3}$

27. $\int \frac{2x+3}{(x^2+3x+5)^4}\,dx$

28. $\int e^{2x}\,dx$

29. $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^3}$

30. $\int x^4 \sin x^5\,dx$

Name:________KEY______
AP Calculus AB: Practice AP Problems

Date:________
Ms. Loughran

1992 AB4/BC1
Solution

(a) $\frac{dy}{dx} - \sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx}(1 - \sin y) = 1$

$\frac{dy}{dx} = \frac{1}{1 - \sin y}$

(b) $\frac{dy}{dx}$ undefined when $\sin y = 1$

$y = \frac{\pi}{2}$

$\frac{\pi}{2} + 0 = x + 1$

$x = \frac{\pi}{2} - 1$

(c) $$\frac{d^2y}{dx^2} = \frac{d\left(\frac{1}{1-\sin y}\right)}{dx}$$

$$= \frac{-\left(-\cos y \frac{dy}{dx}\right)}{(1-\sin y)^2}$$

$$= \frac{\cos y\left(\frac{1}{1-\sin y}\right)}{(1-\sin y)^2}$$

$$= \frac{\cos y}{(1-\sin y)^3}$$

2002 AB 5

(a) When $h = 5$, $r = \frac{5}{2}$; $V(5) = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2 5 = \frac{125}{12}\pi \text{ cm}^3$

(b) $\frac{r}{h} = \frac{5}{10}$, so $r = \frac{1}{2}h$

$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3$; $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$\left.\frac{dV}{dt}\right|_{h=5} = \frac{1}{4}\pi(25)\left(-\frac{3}{10}\right) = -\frac{15}{8}\pi \text{ cm}^3/\text{hr}$

OR

$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$; $\frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$

$\left.\frac{dV}{dt}\right|_{h=5,r=\frac{5}{2}} = \frac{1}{3}\pi\left(\left(\frac{25}{4}\right)\left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right)5\left(-\frac{3}{20}\right)\right)$

$= -\frac{15}{8}\pi \text{ cm}^3/\text{hr}$

(c) $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt} = -\frac{3}{40}\pi h^2$

$= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10}\cdot \text{area}$

The constant of proportionality is $-\frac{3}{10}$.

units of cm^3 in (a) and cm^3/hr in (b)

Scoring:

1 : V when $h = 5$

5 :
- 1 : $r = \frac{1}{2}h$ in (a) or (b)
- 1 : V as a function of one variable in (a) or (b) OR $\frac{dr}{dt}$
- 2 : $\frac{dV}{dt}$ < −2 > chain rule or product rule error
- 1 : evaluation at $h = 5$

2 :
- 1 : shows $\frac{dV}{dt} = k \cdot \text{area}$
- 1 : identifies constant of proportionality

1 : correct units in (a) and (b)

2003 AB 4

(a) The function f is increasing on $[-3, -2]$ since $f' > 0$ for $-3 \le x < -2$.

(b) $x = 0$ and $x = 2$

f' changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$

(c) $f'(0) = -2$

Tangent line is $y = -2x + 3$.

Scoring:

2 :
- 1 : interval
- 1 : reason

2 :
- 1 : $x = 0$ and $x = 2$ only
- 1 : justification

1 : equation

1992 AB 4

(a) $y + \cos y = x+1$

$\frac{dy}{dx} - \sin y \, dy/dx = 1$

$dy/dx(1-\sin y) = 1$

$\frac{dy}{dx} = \frac{1}{1-\sin y}$

b) $1-\sin y = 0$

$1 = \sin y$

$y = \pi/2$

so, $\frac{\pi}{2} + \cos\frac{\pi}{2} = x+1$

$\frac{\pi}{2} + 0 = x+1$

$\frac{\pi}{2} - 1 = x$

(b) $\left[\frac{1}{1-\sin y}\right]' = \frac{-\left(-\cos y \frac{dy}{dx}\right)}{(1-\sin y)^2} = \frac{\cos y \cdot \frac{1}{1-\sin y}}{(1-\sin y)^2}$

$= \frac{\cos y}{(1-\sin y)^3}$

2002 AB5

$\frac{r}{h} = \frac{5}{10} = \frac{1}{2}$

$2r = h$ or $r = \frac{h}{2}$

$\frac{dh}{dt} = -\frac{3}{10}$ cm/hr

a) $V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{3} \cdot \frac{h^3}{4} = \frac{\pi h^3}{12}$

@ $h = 5$ cm $V = \frac{5^3\pi}{12}$ or $\frac{125\pi}{12}$ cm^3

b) $V = \frac{\pi}{12} h^3$

$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{\pi}{4} (5)^2 \cdot \frac{-3}{10}$

$\frac{dV}{dt} = \frac{-75\pi}{40}$ cm^3/hr

c) Need to show that

$\frac{dV}{dt} = \text{constant} \cdot \underset{\pi r^2}{\text{area}}$

*since we are talking about area need to get r back in the mix

$V = \frac{1}{3}\pi r^2 (2r)$

$V = \frac{2}{3}\pi r^3$

$\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 2A \frac{dr}{dt}$

$\frac{dV}{dt} = 2A \cdot \frac{1}{2} \frac{dh}{dt}$

$\frac{dV}{dt} = 2A \cdot \frac{1}{2}\left(\frac{-3}{10}\right)$

$\frac{dV}{dt} = \frac{-3}{20} \cdot 2A$

$\frac{dV}{dt} = \frac{-6}{20} A$

$\frac{-6}{20} = C$

2003 AB4

a) $(-3, -2)$ b/c f' is positive in that interval

b) f has a point of inflection where f'' changes sign, that happens where f' changes from ↗ to ↘ or ↘ to ↗, so at $x = 0, 2$

c) $f'(0) = -2$ $y - 3 = -2(x - 0)$