Name: ________________________________ Date: ______________

AP Calculus AB: Separable Differential Equations Ms. Loughran

Do Now:

1. $\int \frac{\sin x}{\cos^2 x}dx =$ $\int \tan x \sec x \, dx = \sec x + c$

Differential equation: an equation that contain a derivative

We can solve differential equations using a technique called separation of variables. You move different variables to opposite sides of the equation so that you can integrate both sides of the equation separately.

1. Solve for y if $\frac{dy}{dx} = (xy)^2$ and $y = 1$ when $x = 1$.

$$\frac{dy}{dx} = x^2 y^2$$

$$dy = x^2 y^2 dx$$

$$\frac{dy}{y^2} = x^2 dx$$

$$\int y^{-2} dy = \int x^2 dx$$

$$-\frac{1}{y} = \frac{x^3}{3} + c$$

find c (1,1)

$$-\frac{1}{1} = \frac{1^3}{3} + c$$

$$-1 = \frac{1}{3} + c$$

$$-\frac{4}{3} = c$$

$$-\frac{1}{y} = \frac{x^3}{3} - \frac{4}{3}$$

$$-\frac{1}{y} = \frac{x^3-4}{3}$$

$$\frac{1}{y} = \frac{-(x^3-4)}{3} \text{ or } \frac{4-x^3}{3}$$

$$y = \frac{3}{4-x^3}$$

You can check by taking the derivative.

$$\left[\frac{3}{4-x^3}\right]' = 3\left[\frac{1}{4-x^3}\right]' = 3\left[\frac{-(-3x^2)}{(4-x^2)^2}\right]$$

$$\frac{9x^2}{(4-x^3)^2} = x^2 \cdot \frac{9}{(4-x^3)^2}$$

$$x^2 \cdot \left[\frac{3}{4-x^3}\right]^2 = x^2 y^2 = (xy)^2$$

For each of the following, solve for y.

2. $\frac{dy}{dx} = \frac{x}{y}$ and $y = 2$ when $x = 1$.

$$\int x\,dx = \int y\,dy$$

$$\frac{x^2}{2} + C = \frac{y^2}{2}$$

$$\frac{1^2}{2} + C = \frac{2^2}{2}$$

$$\frac{1}{2} + C = 2$$

$$C = \frac{3}{2}$$

(1,2)

$$2\left(\frac{x^2}{2} + \frac{3}{2} = \frac{y^2}{2}\right)$$

$$x^2 + 3 = y^2$$

$$y = \cancel{\pm}\sqrt{x^2+3}$$

reject $\ominus$ b/c graph has to contain the point (1,2)

3. $\frac{dy}{dx} = \frac{y}{x}$ and $y = 2$ when $x = 2$.

$$x\,dy = y\,dx$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + c$$

$$\ln|2| = \ln|2| + c$$

$$0 = c$$

$$\ln|y| = \ln|x| + 0$$

$$\ln|y| = \ln|x|$$

$$e^{\ln|y|} = e^{\ln|x|}$$

$$|y| = |x|$$

$$y = \pm|x|$$

$$y = \cancel{\pm}x$$

reject $\ominus$ b/c it has to contain (2,2)

4. $\frac{dy}{dx} = -2xy^2$ and $y = .25$ when $x = 1$.

$$dy = -2xy^2 dx$$

$$\int \frac{dy}{y^2} = \int -2x\,dx$$

$$-\frac{1}{y} = -x^2 + C$$

$(1, .25)$

$(1, \frac{1}{4})$

$$-\frac{1}{\frac{1}{4}} = -(1)^2 + C$$

$$-4 = -1 + C$$

$$-3 = C$$

$$-\frac{1}{y} = -x^2 - 3$$

$$\frac{1}{y} = x^2 + 3$$

$$y = \frac{1}{x^2+3}$$

5. $\frac{dy}{dx} = (\cos x)e^{y+\sin x}$ and $y = 0$ when $x = 0$.

$$\frac{dy}{dx} = (\cos x)\, e^{y} \cdot e^{\sin x}$$

$$dy = \cos x \cdot e^{y} \cdot e^{\sin x} dx$$

$$\frac{dy}{e^y} = \cos x\, e^{\sin x} dx$$

$$\int e^{-y} dy = \int \cos x\, e^{\sin x} dx$$

$u = \sin x$

$du = \cos x dx$

$\int e^u du$

$e^u + C$

$e^{\sin x} + C$

$$-e^{-y} = e^{\sin x} + C$$

$(0, 0)$

$$-e^{0} = e^{\sin 0} + C$$

$$-1 = 1 + C$$

$$-2 = C$$

$$-e^{-y} = e^{\sin x} - 2$$

$$e^{-y} = -e^{\sin x} + 2$$

$$\ln e^{-y} = \ln(-e^{\sin x} + 2)$$

$$-y = \ln(-e^{\sin x} + 2)$$

$y = -\ln(-e^{\sin x} + 2)$,

$-e^{\sin x} + 2 > 0$

$2 > e^{\sin x}$

$e^{\sin x} < 2$

6. $\frac{dy}{dx} = (y+5)(x+2)$ and $y = 1$ when $x = 0$.

$\int \frac{dy}{y+5} = \int (x+2)\,dx$

$\ln|y+5| = \frac{x^2}{2} + 2x + c$

(0,1) ← find c first then we isolate y

$\ln|1+5| = \frac{0^2}{2} + 2(0) + c$

$\ln 6 = c$

$\ln|y+5| = \frac{x^2}{2} + 2x + \ln 6$

$e^{\ln|y+5|} = e^{\frac{x^2}{2} + 2x + \ln 6}$

$|y+5| = e^{\frac{x^2}{2} + 2x} \cdot e^{\ln 6}$

$|y+5| = 6e^{\frac{x^2}{2} + 2x}$

$y+5 = \pm 6e^{\frac{x^2}{2} + 2x}$

$y = \cancel{\pm} 6e^{\frac{x^2}{2} + 2x} - 5$

b/c it has to contain (0,1)

other method

isolate y first then find c

$\ln|y+5| = \frac{x^2}{2} + 2x + c$

$e^{\ln|y+5|} = e^{\frac{x^2}{2} + 2x + c}$

$|y+5| = e^{\frac{x^2}{2} + 2x} \cdot e^{c}$ → this is just a new constant

$|y+5| = ke^{\frac{x^2}{2} + 2x}$

$y+5 = \pm ke^{\frac{x^2}{2} + 2x}$

$y+5 = k_2 e^{\frac{x^2}{2} + 2x}$

$y = k_2 e^{\frac{x^2}{2} + 2x} - 5$

then plug in (0,1)

$1 = k_2 e^0 - 5$

$1 = k_2 - 5$

$6 = k_2$

$y = 6e^{\frac{x^2}{2} + 2x} - 5$

Homework 02-15

$$\begin{array}{r|l} & 1 \\ \hline x^2+4 & x^2 \\ & x^2+4 \\ \hline & -4 \end{array}$$

$1+\frac{-4}{x^2+4}$

1. $\int \frac{x^2}{x^2+4}dx$

$$\int\left(1+\frac{-4}{x^2+4}\right)dx = \int 1dx - 4\int\frac{1}{x^2+4}dx = x - 4\int\frac{1}{\underset{\textcircled{4}(1+\frac{x^2}{4})}{4+x^2}} = x-4\cdot\frac{1}{4}\int\frac{1}{1+\frac{x^2}{4}}$$

$u=\frac{x}{2}$

$du=\frac{1}{2}dx$

$2du=dx$

$$= x-2\int\frac{1}{1+u^2}$$

$$= x - 2\arctan\left(\frac{x}{2}\right)+C$$

2. $\int \frac{1}{x^2+6x+14}dx$

$x^2+6x+9-9+14$

$(x+3)^2+5$

$$\int\frac{1}{(x+3)^2+5}dx$$

$$\frac{1}{5}\int\frac{1}{\frac{(x+3)^2}{5}+1}dx$$

$$\frac{1}{5}\int\frac{1}{\left(\frac{x+3}{\sqrt{5}}\right)^2+1}$$

$u=\frac{x+3}{\sqrt{5}}$

$du=\frac{1}{\sqrt{5}}dx$

$\sqrt{5}\,du=dx$

$$\frac{1}{5}\cdot\sqrt{5}\int\frac{1}{u^2+1}du$$

$$\frac{\sqrt{5}}{5}\tan^{-1}u+C$$

$$\frac{\sqrt{5}}{5}\tan^{-1}\left(\frac{x+3}{\sqrt{5}}\right)+C$$

3. $\int_0^1 \frac{x^3}{2x^2+1}dx$

$$\begin{array}{r|l} & \frac{1}{2}x \\ \hline 2x^2+1 & x^3 \\ & x^3+\frac{1}{2}x \\ \hline & -\frac{1}{2}x \end{array}$$

$$\int_0^1 \frac{1}{2}x - \frac{\frac{1}{2}x}{2x^2+1}dx = \int_0^1 \frac{1}{2}x - \frac{x}{\underset{4(x^2+\frac{1}{2})}{4x^2+2}}dx$$

$$= \frac{1}{2}\frac{x^2}{2} - \int_0^1\frac{x}{4x^2+2}dx$$

$$\left.\frac{x^2}{4}\right|_0^1$$

$u=4x^2+2$

$du=8x\,dx$

$\frac{du}{8}=x\,dx$

$$\left.\frac{x^2}{4}\right|_0^1 - \frac{1}{8}\int_2^6 u^{-1}du$$

$$\frac{1}{4}-0-\frac{1}{8}\ln|u|\Big|_2^6$$

$$\frac{1}{4}-0-\frac{1}{8}\ln 6+\frac{1}{8}\ln 2$$

$$\frac{1}{4}-\frac{1}{8}\ln\left(\frac{6}{2}\right) = \frac{1}{4}-\frac{1}{8}\ln 3$$

④
$$\begin{array}{r|l} & 1 \\ \hline x-3 & x+6 \\ & x-3 \\ \hline & 9 \end{array}$$

$$\int 1+\frac{9}{x-3}dx$$

$$x+9\ln|x-3|+C$$

⑤ $\int \frac{dv}{(x+1)^2+1}$ $\quad$ $u=x+1$ $\quad$ $du=dx$ $\longrightarrow$ how?

x^2+2x+2

$x^2+2x+1-1+2$

$(x+1)^2+1$

$\int \frac{1}{u^2+1}\,du = \tan^{-1}u + C$ $\quad$ 1

$\tan^{-1}(x+1)+C$

⑥

$-2x^2-8x-6$

$-2(x^2+4x+4-4\ )-6$

$-2(x+2)^2+8-6$

$-2(x+2)^2+2 \qquad 2-2(x+2)^2$

$$\int \frac{1}{\sqrt{2-2(x+2)^2}}\,dx$$

$$\int \frac{1}{\sqrt{2(1-(x+2)^2)}}\,dx \qquad \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{1-(x+2)^2}} = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$u=x+2$

$du=dx$

$\frac{1}{\sqrt{2}} \cdot \sin^{-1}u + C$

$\frac{1}{\sqrt{2}}\sin^{-1}(x+2)+C$

⑦ $\int_3^4 \frac{dx}{x^2-6x+9-9+10}$

$$\int_3^4 \frac{dx}{(x-3)^2+1} \qquad u=x-3 \quad du=dx \qquad \int_0^1 \frac{1}{u^2+1}\,du = \tan^{-1}u\Big|_0^1$$

$\tan^{-1}1-\tan^{-1}0$

$\frac{\pi}{4}-0=\frac{\pi}{4}$

Classwork 02-16

PREVIOUS ANSWER:

$\sin^{-1}(x+7)+c$

$$\int \frac{x^3-1}{x-2}\,dx =$$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 0 & -1 \\ & & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 7 \end{array}$$

$$\int\left(x^2+2x+4+\frac{7}{x-2}\right)dx$$

$$\frac{x^3}{3}+x^2+4x+7\ln|x-2|+c$$

PREVIOUS ANSWER:

$\frac{x^3}{3}+x^2+4x+7\ln|x-2|+c$

$$\int \frac{1}{5x^2-20x+100}\,dx=$$

$5(x^2-4x+20)$

$$\frac{1}{5}\int \frac{1}{x^2-4x+20}\,dx$$

$x^2-4x+4-4+20$

$(x-2)^2+16$

$$\frac{1}{5}\int \frac{1}{(x-2)^2+16}\,dx$$

$$\frac{1}{5}\int \frac{1}{16\left(\left(\frac{x-2}{4}\right)^2+1\right)}\,dx$$

$u=\frac{x-2}{4}$

$du=\frac{1}{4}\,dx$

$4du=dx$

$$4\cdot\frac{1}{80}\int \frac{1}{u^2+1}\,)=\frac{1}{20}\tan^{-1}\left(\frac{x-2}{4}\right)+c$$

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PREVIOUS ANSWER:
$\frac{1}{20}\arctan\left(\frac{x-2}{4}\right)+c$

Given f is a linear function and $0 < a < b$, find $\int_a^b f''(x)dx$.

$f'(x) \rightarrow$ slope

$f'(b) - f'(a)$

$m - m = 0$

PREVIOUS ANSWER: 0

If $f(x) = \begin{cases} x & \text{for } x \le 1 \\ \dfrac{1}{x} & \text{for } x > 1, \end{cases}$ find $\displaystyle\int_0^e f(x)\,dx$.

$$\int_0^1 x\,dx + \int_1^e \frac{1}{x}\,dx$$

$$\left.\frac{x^2}{2}\right|_0^1 + \ln|x|\Big|_1^e$$

$$\frac{1}{2} + \ln e - \ln 1 \quad = \frac{1}{2} + \ln e$$

$$= \frac{1}{2} + 1 = \frac{3}{2}$$

PREVIOUS ANSWER: $\frac{3}{2}$

If $\int_1^4 f(x)dx = 10$,

find $\int_1^4 (2f(x) + 5)dx$.

$$2\int_1^4 f(x)dx + \int_1^4 5dx$$

$$2(10) + 5x\Big|_1^4$$

$$20 + 20 - 5 = 35$$

PREVIOUS ANSWER: 35

$$\int_0^2 (10x^4 + 3x^2 - 25)dx$$

$$\frac{10x^5}{5} + \frac{3x^3}{3} - 25x \Big|_0^2$$

$$2x^5 + x^3 - 25x \Big|_0^2$$

$$2(2)^5 + 2^3 - 25(2) - 0$$

$$64 + 8 - 50$$

$$22$$

PREVIOUS ANSWER: 22

$$\int \frac{x^2}{x+5}\,dx$$

$$\begin{array}{r|rrr} -5 & 1 & 0 & 0 \\ & & -5 & 25 \\ \hline & 1 & -5 & 25 \end{array}$$

$$\int \left(x-5+\frac{25}{x+5}\right)dx$$

$$\frac{x^2}{2}-5x+25\ln|x+5|+C$$

PREVIOUS ANSWER:

$\frac{1}{2}x^2 - 5x + 25\ln|x+5| + C$

$$\int 16x\sin(x^2)\,dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$16 \cdot \frac{1}{2}\int \sin u\,du$

$-8\cos u + C$

$-8\cos(x^2) + C$

PREVIOUS ANSWER:
$-8\cos(x^2) + C$

$$\int_0^{\frac{\pi}{2}} \sin(x)\, e^{\cos(x)}\, dx$$

$u = \cos x$

$du = -\sin x dx$

$-du = \sin x dx$

$$-\int_1^0 e^u du = \int_0^1 e^u du$$

$$= e^u \Big|_0^1 = e^1 - e^0$$

$e - 1$

PREVIOUS ANSWER: $e - 1$

$$\int \frac{8}{\sec(x)} dx$$

$$8\int \cos x \, dx$$

$$8\sin x + c$$

PREVIOUS ANSWER:

$8\sin(x) + C$

$$\int x(x-7)^3 dx$$

$u = x-7 \quad \Rightarrow \quad x = u+7$

$du = dx$

$$\int (u^3(u+7))du$$

$$\int (u^4 + 7u^3)du$$

$$\frac{u^5}{5} + \frac{7u^4}{4} + C = \frac{(x-7)^5}{5} + \frac{7(x-7)^4}{4} + C$$

PREVIOUS ANSWER:

$$\frac{(x-7)^5}{5}+\frac{7(x-7)^4}{4}+C$$

If $\int_0^7 f(x)dx = 3$ and

$\int_7^0 g(x)dx = 12$,

find $\int_0^7 \left(f(x) - \frac{1}{2}g(x)\right)dx$.

$$\int_0^7 f(x)dx - \frac{1}{2}\int_0^7 g(x)\,dx$$

$$3 - \frac{1}{2}(-12) = 3+6 = 9$$

PREVIOUS ANSWER: 9

$$\int (x^5 - 2x + 7)dx$$

$$\frac{x^6}{6} - \frac{2x^2}{2} + 7x + C$$

$$\frac{1}{6}x^6 - x^2 + 7x + C$$

PREVIOUS ANSWER:

$\frac{1}{6}x^6 - x^2 + 7x + C$

$$\int_1^{e^2} \frac{\ln(x)}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$\int_0^2 u\,du = \frac{u^2}{2}\Big|_0^2$$

$$\frac{2^2}{2} - 0 = 2$$

PREVIOUS ANSWER: 2

$$\int \frac{2\cos(x)}{\sin^3(x)}\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$2\int u^{-3}\,du = \cancel{2}\cdot\frac{u^{-2}}{-\cancel{2}} + C$$

$$-\frac{1}{u^2} + C$$

$$-\frac{1}{\sin^2 x} + C$$

$$-\csc^2 x + C$$

PREVIOUS ANSWER:

$-\csc^2 x + c$

$$\int 12e^{3x-5}\,dx$$

$12 \cdot \frac{1}{3} e^{3x-5} + C$

$4e^{3x-5} + C$

PREVIOUS ANSWER:

$4e^{3x-5} + C$

$$\int_0^{\frac{\pi}{4}} \sec^2(x)\, dx$$

$$\tan x \Big|_0^{\frac{\pi}{4}} = \tan \tfrac{\pi}{4} - \tan 0$$

$$1 - 0 = 1$$

PREVIOUS ANSWER: 1

$$\int \frac{x-5}{x} dx$$

$$\int \left(1 - \frac{5}{x}\right) dx$$

$$x - 5 \ln|x| + C$$

PREVIOUS ANSWER:

$x - 5\ln|x| + C$

If $\int_0^a (x-3)dx = 8$ and $a > 0$, find the value of a.

$$\frac{x^2}{2} - 3x \Big|_0^a = 8$$

$$\frac{a^2}{2} - 3a = 8$$

$$a^2 - 6a = 16$$

$$a^2 - 6a - 16 = 0$$

$$(a-8)(a+2) = 0$$

$a = 8$

$\cancel{a = -2}$ b/c

PREVIOUS ANSWER: 8

$$\int \frac{12}{3x-5}dx$$

$$12\int \frac{1}{3x-5}\,dx$$

$u = 3x - 5$

$du = 3dx$

$\frac{du}{3} = dx$

$12 \cdot \frac{1}{3} \int \frac{1}{u}\,du = 4\ \ln|u| + c$

$4\ \ln|3x-5| + c$

PREVIOUS ANSWER:

$4\ln|3x-5|+C$

$$\int \frac{1}{\sqrt{-x^2-14x-48}}\,dx =$$

$-(x^2+14x+49-49+48)$

$-(x+7)^2+1$

$$\int \frac{1}{\sqrt{1-(x+7)^2}}\,dx$$

$u=x+7$

$du=dx$

$$\int \frac{1}{\sqrt{1-u^2}}\,du$$

$\arcsin u + C$

$\arcsin(x+7)+C$