Name: ______________________________ Date: __________

AP Calc: Relationship Between Area under the curve and the Definite Integral

Do Now:

1. $\int_0^1 x^2 dx =$ $\left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$ ← total area

2. $\int_{-\pi}^{\frac{\pi}{3}} \sin x dx =$ $\left.-\cos x\right|_{-\pi}^{\frac{\pi}{3}} = -\cos\frac{\pi}{3} - (-\cos(-\pi))$

$-\frac{1}{2} - (1) = \frac{-3}{2}$ ← net area

$-(\cos\frac{\pi}{3} - \cos(-\pi))$

Wrapping up from yesterday…

**Trapezoid Rule**

When you use the trapezoid rule, you get the average of the left and right hand approximation for the area under the curve. Let's prove it!

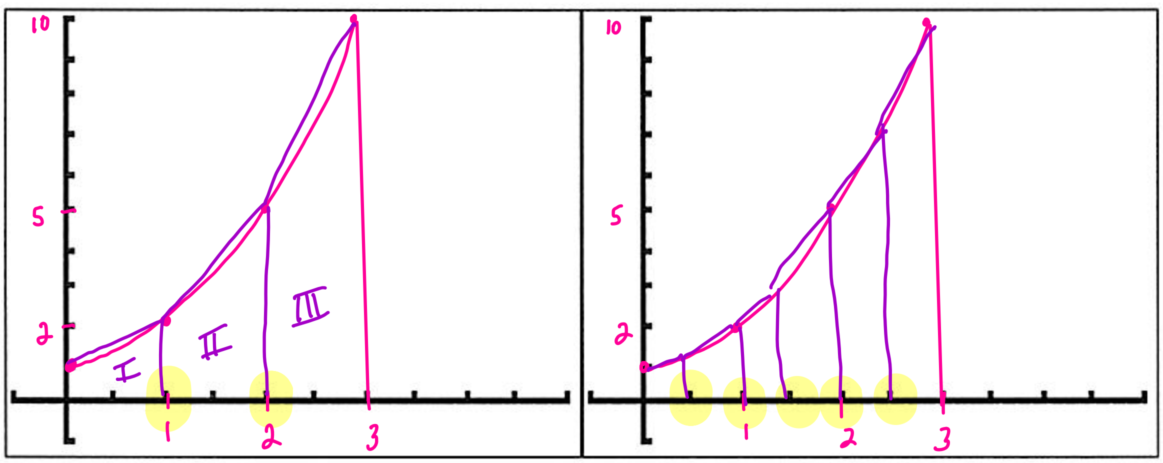
Example: Graph $f(x) = x^2 + 1 \quad 0 \le x \le 3$

To make the trapezoids, first draw a line from each interval endpoint to the curve.
Our intervals are: (0, 1) (1, 2) and (2, 3), so the endpoints of these intervals are 1, 2, and 3.
Now the bottom of all of the trapezoids is the x axis. To get the top of each trapezoid connect the y values of each interval.

Graph with 3 trapezoids $h = \frac{3-0}{3} = 1$

Graph with 6 trapezoids $h = \frac{3-0}{6} = \frac{1}{2}$

| x | y |
|---|---|
| 0 | 1 |
| .5 | 1.25 |
| 1 | 2 |
| 1.5 | 3.25 |
| 2 | 5 |
| 2.5 | 7.25 |
| 3 | 10 |




Area of a trapezoid is: $A = \frac{1}{2}(b_1 + b_2)h$

The b values in the formula are the parallel bases and h is the height (we can also say the height is the length between each trapezoid)

Calculations:

Area with 3 Trapezoids

$A = A_{I} + A_{II} + A_{III}$

$\frac{1}{2}(1)(1+2) + \frac{1}{2}(1)(2+5) + \frac{1}{2}(1)(5+10)$

$A = \frac{1}{2}(1)\left(1 + 2(2) + 2(5) + 10\right)$

Area with 6 Trapezoids

$A = \frac{1}{2}\left(\frac{1}{2}\right)\left(1 + 2(1.25) + 2(2) + 2(3.25) + 2(5) + 2(7.25) + 10\right)$

positive

If a function $f$ is continuous on $[a,b]$ and if $f(x) \geq 0$ for all $x$ in $[a,b]$ then the area under the curve $y = f(x)$ over the interval $[a,b]$ is defined by:

$$Area = \lim_{n \to +\infty} \sum_{k=1}^{n} \underbrace{f(x_k)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}$$

Which can be rewritten as : $$Area = \int_a^b f(x)dx$$

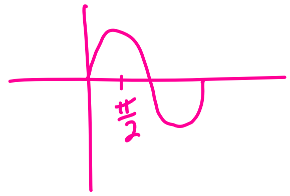
Recall the FTC Part I:

If a function is continuous on $[a,b]$

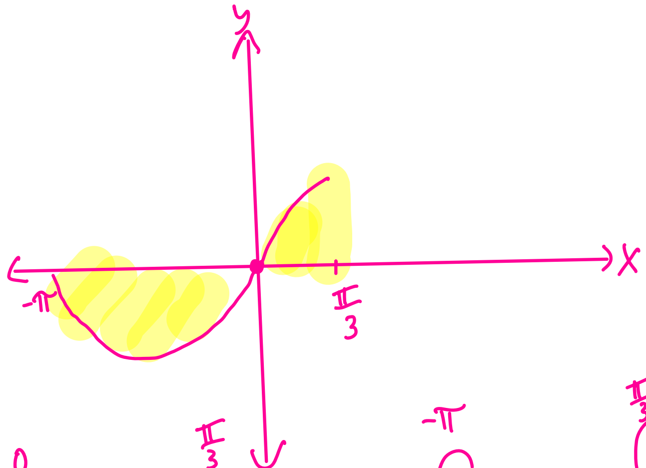
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Corollary: If $f(x) < 0$ on $[a,b]$ then

$$Area = -\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$$




Let's find the total area bounded by the curve $y = \sin x$ and the $x$-axis from $\left[-\pi, \frac{\pi}{3}\right]$.




$$-\int_{-\pi}^{0} \sin x\,dx + \int_{0}^{\frac{\pi}{3}} \sin x\,dx = \int_{0}^{-\pi} \sin x\,dx + \int_{0}^{\frac{\pi}{3}} \sin x\,dx$$

$$-\cos x\Big|_{0}^{-\pi} - \cos x\Big|_{0}^{\frac{\pi}{3}}$$

$$-\left(\cos(-\pi) - \cos 0\right) - \left(\cos\frac{\pi}{3} - \cos 0\right)$$

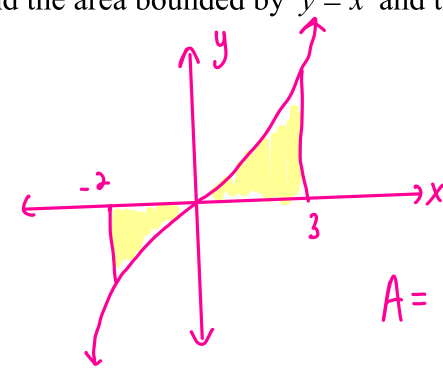
$$-(-1-1) - \left(\frac{1}{2} - 1\right) = 2 + \frac{1}{2} = \frac{5}{2}$$

Examples:

1. Find the value of $\int_{-2}^{3} x^3\,dx =$

$$\frac{x^4}{4}\Big|_{-2}^{3} = \frac{81}{4} - \left(\frac{16}{4}\right) = \frac{65}{4} \leftarrow \text{net area}$$

2. Find the area bounded by $y = x^3$ and the $x$-axis from $x = -2$ to $x = 3$.




$$A = -\int_{-2}^{0} x^3\,dx + \int_{0}^{3} x^3\,dx$$

$$\int_{0}^{-2} x^3\,dx + \int_{0}^{3} x^3\,dx = \frac{x^4}{4}\Big|_{0}^{-2} + \frac{x^4}{4}\Big|_{0}^{3}$$

$$= \frac{16}{4} + \frac{81}{4} = \frac{97}{4}$$

$(-x)^3 - 4(-x) = -x^3 + 4x$

$-(x^3 - 4x)$ odd

3. Find the value of $\int_{-2}^{2} \underbrace{(x^3 - 4x)}_{}^{\text{odd}} dx = 0$

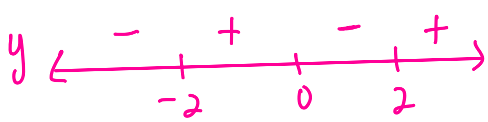
4. Find the area bounded by $y = x^3 - 4x$ and the $x$-axis from $x = -2$ to $x = 2$.

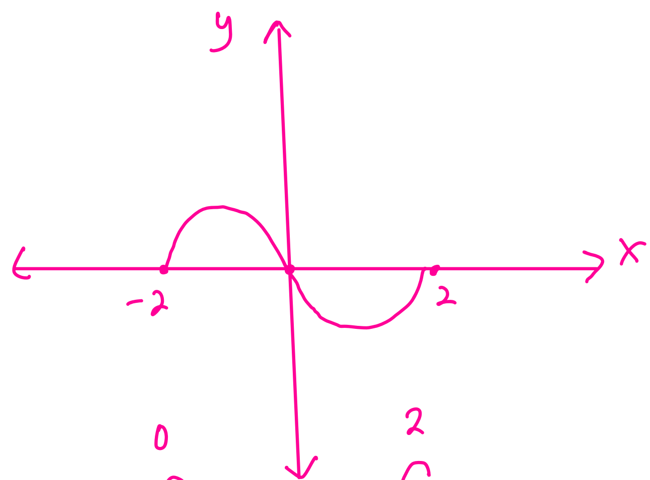
$y = x^3 - 4x$

$y = x(x^2 - 4)$

$0 = x(x^2 - 4) = x(x-2)(x+2)$

$x = 0, \pm 2$

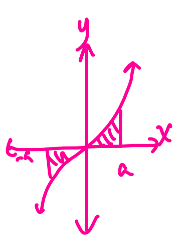


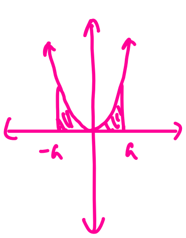

$A = \int_{-2}^{0} (x^3 - 4x)\,dx - \int_{0}^{2} (x^3 - 4x)\,dx$

$\int_{-2}^{0} (x^3 - 4x)\,dx + \int_{2}^{0} (x^3 - 4x)\,dx = 8$

A note about evaluating integrals:



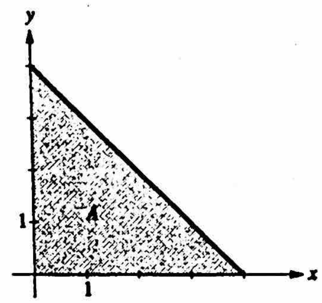

$\int_{-a}^{a}$ odd function $= 0$




$\int_{-a}^{a}$ even function $= 2\int_{-a}^{0}$ function or $2\int_{0}^{a}$ function

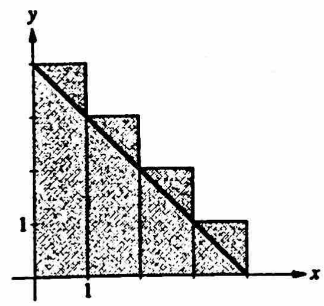
Homework 02-28

1. (a) $\frac{1}{2}bh = \frac{1}{2}\cdot 4\cdot 4 = 8$



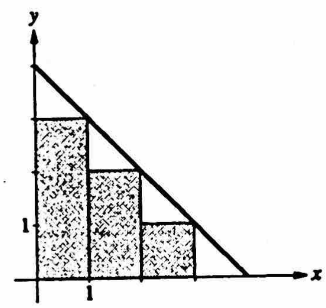
(b) The approximation is greater than $A$, as the rectangles extend beyond the area.

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = (4+3+2+1)(1) = 10$$



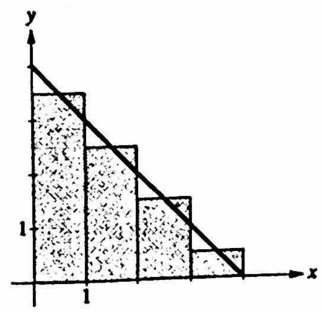
(c) The approximation is less than $A$, as the rectangles lie inside the area.

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = (3+2+1+0)(1) = 6$$



(d) The approximation is equal to $A$, as can be seen by measuring congruent triangles.

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = (3.5+2.5+1.5+0.5)(1) = 8$$



**3.** (a) $x_k^* = 0, 1, 2, 3, 4$

$$\sum_{k=1}^{5} f(x_k^*)\Delta x = (1 + 2 + 5 + 10 + 17)(1) = 35$$

(b) $x_k^* = 1, 2, 3, 4, 5$

$$\sum_{k=1}^{5} f(x_k^*)\Delta x = (2 + 5 + 10 + 17 + 26)(1) = 60$$

(c) $x_k^* = 1/2, 3/2, 5/2, 7/2, 9/2$

$$\sum_{k=1}^{5} f(x_k^*)\Delta x = (5/4 + 13/4 + 29/4 + 53/4 + 85/4)(1) = 185/4 = 46.25$$

**4.** (a) $x_k^* = 1, 2, 3, 4, 5$

$$\sum_{k=1}^{5} f(x_k^*)\Delta x = (1 + 8 + 27 + 64 + 125)(1) = 225$$

(b) $x_k^* = 2, 3, 4, 5, 6$

$$\sum_{k=1}^{5} f(x_k^*)\Delta x = (8 + 27 + 64 + 125 + 216)(1) = 440$$

(c) $x_k^* = 3/2, 5/2, 7/2, 9/2, 11/2$

$$\sum_{k=1}^{5} f(x_k^*)\Delta x = (27/8 + 125/8 + 343/8 + 729/8 + 1331/8)(1) = 2555/8 = 319.375$$