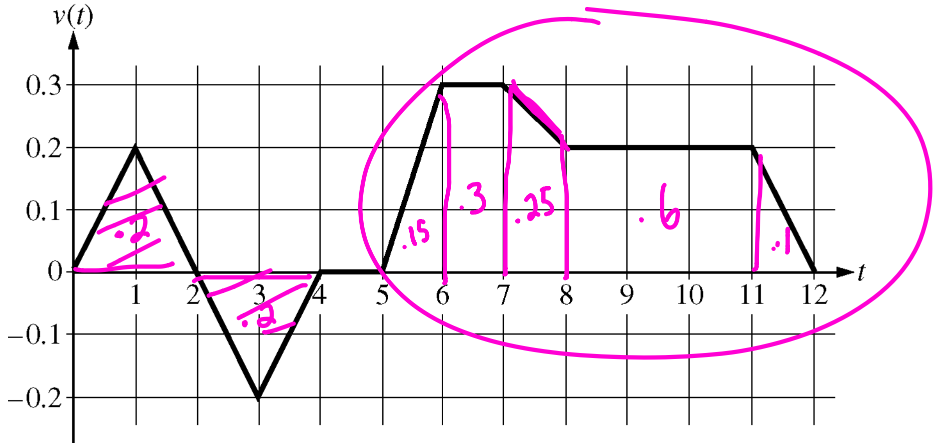
Do Now:

CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

   (a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

   $\dfrac{0.3-0.2}{7-8} = -.1$ miles/min^2

   (b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

   The total distance in miles that Caren has covered from 0 to to 12 min. $\int_0^{12} |v(t)|\,dt = .2 + .2 + .15 + .3 + .25 + .6 + .1$ miles $= 1.8$ miles

   (c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

   (d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15} \sin\left(\frac{\pi}{12} t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

WRITE ALL WORK IN THE PINK EXAM BOOKLET.

c) At 2 minutes b/c velocity changes at $t=2$

d) $\int_0^{12} |w(t)|\,dt = 1.6$ miles

Caren's distance from home $1.8 - .4$
1.4

GO ON TO THE NEXT PAGE.

$$\frac{1}{b-a}\int_a^b v(t)\,dt = \frac{x(b)-x(a)}{b-a}$$

$$\frac{1}{b-a}\int_a^b a(t)\,dt = \frac{v(b)-v(a)}{b-a}$$

Name:______________________________ Date:__________
AP Calculus: The Definite Integral as an Average Ms. Loughran

We know how to find the average of n numbers: add them and divide by n. But how do we find the average value of a continuously varying function? Let us consider an example. Suppose $C = f(t)$ is the temperature at time t, measured in hours since midnight, and that we want to calculate the average temperature over a 24-hour period. One way to start would be to average the temperatures at n times, $t_1, t_2, \ldots, t_n$, during the day.

$$\text{Average temperature} \approx \frac{f(t_1) + f(t_2) + \cdots + f(t_n)}{n}.$$

The larger we make n, the better the approximation. We can rewrite this expression as a Riemann sum over the interval $0 \le t \le 24$ if we use the fact that $\Delta t = 24/n$, so $n = 24/\Delta t$:

$$\begin{aligned}\text{Average temperature} &\approx \frac{f(t_1) + f(t_2) + \cdots + f(t_n)}{24/\Delta t} \\ &= \frac{f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t}{24} \\ &= \frac{1}{24}\sum_{i=1}^{n} f(t_i)\Delta t.\end{aligned}$$

As $n \to \infty$, the Riemann sum tends towards an integral and also approximates the average temperature better. Thus, in the limit

$$\begin{aligned}\text{Average temperature} &= \lim_{n\to\infty} \frac{1}{24}\sum_{i=1}^{n} f(t_i)\Delta t \\ &= \frac{1}{24}\int_0^{24} f(t)\,dt.\end{aligned}$$

Thus we have found a way of expressing the average temperature in terms of an integral. Generalizing for any function f, we define

$$\boxed{\begin{array}{c}\text{Average value of } f \\ \text{from } a \text{ to } b\end{array} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

How to Visualize the Average on a Graph

The definition of average value tells us that

$$(\text{Average value of } f)\cdot(b-a) = \int_a^b f(x)\,dx.$$

Thus, if we interpret the integral as the area under the graph of f, then we can think of the average value of f as the height of the rectangle with the same area that is on the same base, $(b-a)$. (See Figure 3.18.)

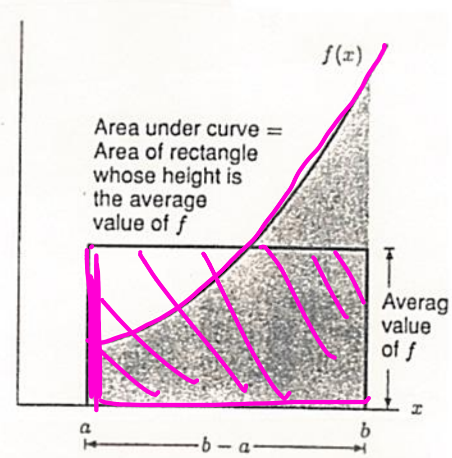


Figure 3.18: Area and average value

Practice

1. Find the average value of $f(x)=x^2$ from $x=2$ to $x=4$.

$$\frac{1}{4-2}\int_2^4 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\Big|_2^4\right] = \frac{1}{2}\left[\frac{64}{3}-\frac{8}{3}\right]$$

$$= \frac{1}{2}\left(\frac{56}{3}\right) = \frac{28}{3}$$

2. Find the average value of $f(x)=\sqrt{x}$ from $x=0$ to $x=16$.

$$\frac{1}{16-0}\int_0^{16}\sqrt{x}\,dx = \frac{1}{16}\left[\frac{2}{3}x^{3/2}\Big|_0^{16}\right] = \frac{1}{24}\left[64-0\right]$$

$$= \frac{64}{24} = \frac{8}{3}$$

3. Find the average value of $f(x)=e^{2x}$ on the interval [-1,1].

$$\frac{1}{2}\int_{-1}^{1} e^{2x}\,dx = \frac{1}{2}\left[\frac{1}{2}e^{2x}\Big|_{-1}^{1}\right]$$

$$\frac{1}{4}\left[e^{2}-(e^{-2})\right]$$

4. Find the average value of $f(x)=\cos x$ from $x=0$ to $x=\pi$.

$$\frac{1}{\pi}\int_0^{\pi}\cos x\,dx = \frac{1}{\pi}\left[\sin x\Big|_0^{\pi}\right] = \frac{1}{\pi}\left[0-0\right] = 0$$

5. Find the average velocity of $v(t)=t^2-2$ on the interval [-2,3].

$$\frac{1}{5}\int_{-2}^{3}(t^2-2)\,dt = \frac{1}{5}\left[\frac{t^3}{3}-2t\Big|_{-2}^{3}\right]$$

$$= \frac{1}{3}$$

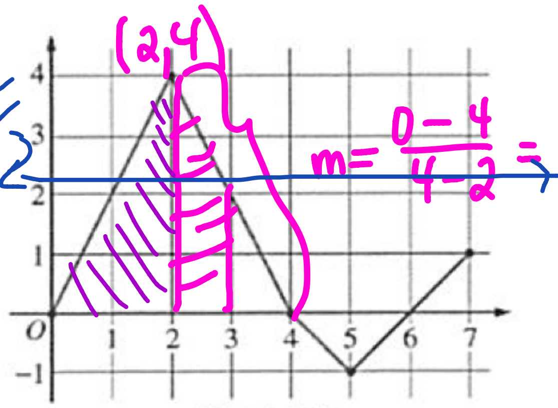
2003 AB 5 Form B (Skip b and c)

$g(0) = \int_2^0 f(t)\,dt$

$-\int_0^2 f(t)\,dt = -4$

$y = \frac{7}{3}$

$m = \frac{0-4}{4-2} = -2$



Graph of f

$g'(x) = f(x)$

Let f be a function defined on the closed interval $[0, 7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

g'

b) $\frac{1}{3-0}\int_0^3 g'(x)\,dx = \frac{1}{3}\left[g(3) - g(0)\right]$

$\frac{1}{3}\left[3 + 4\right] = \frac{7}{3}$

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning.

$g'(x) = 7/3$ two

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(1)(2+4) = 3$

$g'(3) = f(3) = 2$

$g''(3) = -2$

d) $x = 2, 5$ b/c f' changes from ↗ to ↘ or ↘ to ↗ around those x values.

Homework 03-01

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$h(x)$
1	6	4	2	5	3
2	9	2	3	1	4
3	10	–4	4	2	–7
4	–1	3	6	7	

3. The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

   (a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

   (b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

   (c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

   $w'(x) = f(g(x)) \cdot g'(x)$ $\quad w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot g'(3) = -1 \cdot 2 = -2$

   (d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

   $g^{-1}(2) = 1 \leftarrow$ point

   need $(g^{-1})'(2) = \dfrac{1}{g'(1)} = \dfrac{1}{5}$

WRITE ALL WORK IN THE PINK EXAM BOOKLET.

$y - 1 = \frac{1}{5}(x - 2)$

END OF PART A OF SECTION II

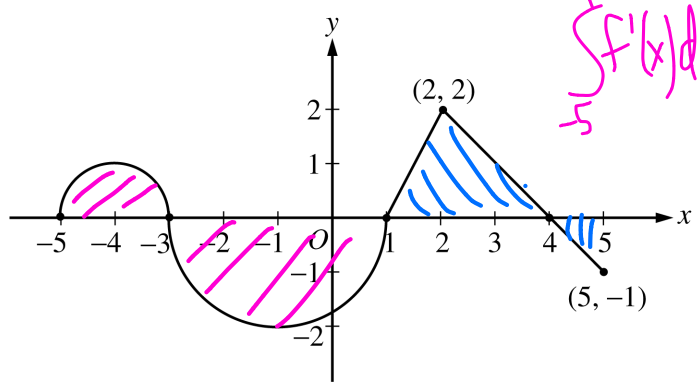
a) Since $h(1) = 3$ and $h(3) = -7$ and $h(x)$ is continuous, by the IVT there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$

Since h is differentiable, by the MVT there exists a c, $1 < c < 3$, such that $h'(c) = -5$.

CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



Graph of f'

$\int_{-5}^{1} f'(x)\,dx = \frac{1}{2}\pi(1)^2 - \frac{1}{2}\pi(2)^2$

$\frac{\pi}{2} - 2\pi$

$\int_{1}^{5} f'(x)\,dx = \frac{1}{2}(3)(2) - \frac{1}{2}(1)(1)$

$= 3 - \frac{1}{2}$

4. Let f be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of f', the derivative of f, consists of two semicircles and two line segments, as shown above.

(a) For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

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a) f has a rel min where f' changes from + to − so at $x = -3, 4$

b) f has a point of inflection where f'' changes sign from + to − or − to +. f' changes from ↗ to ↘ or ↘ to ↗ where f'' changes sign so at $x = -4, -1, 2$

c) f is CU where f'' is positive, f' ↗ and positive so from $(-5,-4)$, $(1,2)$

d) The abs. minimum of f has to occur at a rel min or an endpt

$f(-5) = 3 - \frac{\pi}{2} + 2\pi$

$f(1) = 3$

$f(5) = 6 - \frac{1}{2}$

$\int_{-5}^{1} f'(x)\,dx = f(1) - f(-5)$

$\frac{\pi}{2} - 2\pi = 3 - f(-5)$

$f(-5) = 3 - \frac{\pi}{2} + 2\pi$

$\int_{1}^{5} f'(x)\,dx = f(5) - f(1)$

$3 - \frac{1}{2} = f(5) - 3$

$6 - \frac{1}{2} = f(5)$

∴ Abs min is 3

WRITE ALL WORK IN THE EXAM BOOKLET.

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$\int_{-3}^{2} g'(x) = g(2) - g(-3)$

$-\frac{1}{2}(4)(2)+\frac{1}{2}(3)(1) = 5 - g(-3)$

$-4+\frac{3}{2} = 5 - g(-3)$

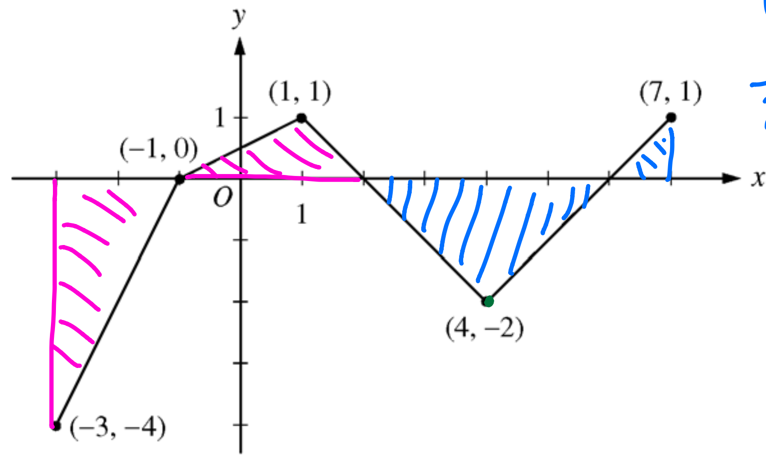
$g(-3) = 5+4-\frac{3}{2}$

$\int_{2}^{7} g'(x)\,dx = g(7) - g(2)$

$-\frac{1}{2}(4)(\cancel{2})+\frac{1}{2}(1)(1) = g(7) - 5$

$-4+\frac{1}{2}+5 = g(7)$

$1+\frac{1}{2} = g(7)$



Graph of g'

5. Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

(a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

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a) $g(x)$ has a point of inflection where g'' changes sign. g'' changes sign where g' changes from ↗ to ↘ or ↘ to ↗ so $x = 1, 4$.

b) By the candidate test, an absolute maximum occurs at an endpoint or relative maximum

$g(-3) = 5+4-\frac{3}{2}$

$g(2) = 5$

$g(7) = 1+\frac{1}{2}$

$\therefore$ abs max is $5+4-\frac{3}{2}$

c) $\dfrac{g(7)-g(-3)}{7-(-3)} = \dfrac{\frac{3}{2}-\left(9-\frac{3}{2}\right)}{10} = \dfrac{\frac{3}{2}-\frac{15}{2}}{10} = \dfrac{\frac{-12}{2}}{10} = \dfrac{-6}{10}$

d) $\dfrac{g'(7)-g'(-3)}{7-(-3)} = \dfrac{1-(-4)}{10} = \dfrac{5}{10} = \dfrac{1}{2}$

No the MVT does not apply here because $g'(x)$ is not differentiable on $(-3,7)$