

Name: _____

Date: _____

AP Calc: Relationship Between Area under the curve and the Definite Integral

Do Now:

$$1. \int_0^1 x^2 dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$$2. \int_{-\pi}^{\frac{\pi}{3}} \sin x dx = \left. -\cos x \right|_{-\pi}^{\frac{\pi}{3}}$$
$$-\left(\cos\left(\frac{\pi}{3}\right) - \cos(-\pi) \right)$$
$$-\left(\frac{1}{2} - (-1) \right)$$
$$-\frac{3}{2} \text{ } \leftarrow \text{ net Area}$$

Wrapping up from Friday...

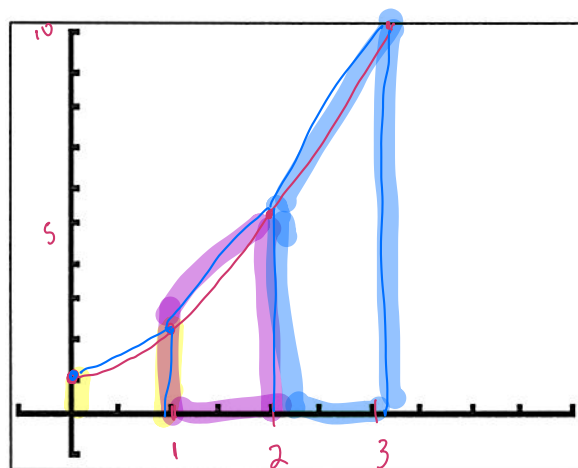
Trapezoid Rule

When you use the trapezoid rule, you get the average of the left and right hand approximation for the area under the curve. Let's prove it!

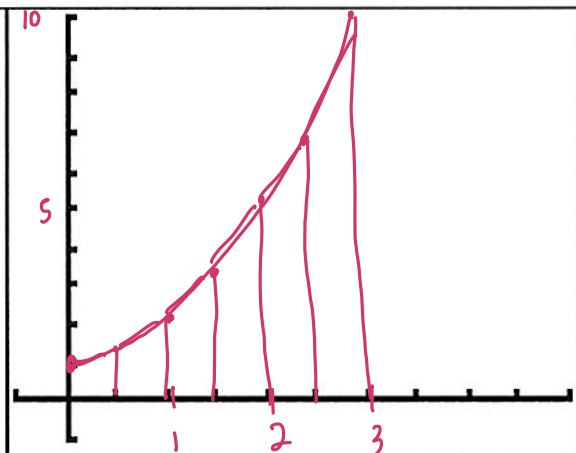
Example: Graph $f(x) = x^2 + 1$ $0 \leq x \leq 3$

To make the trapezoids, first draw a line from each interval endpoint to the curve. Our intervals are: (0, 1) (1, 2) and (2, 3), so the endpoints of these intervals are 1, 2, and 3. Now the bottom of all of the trapezoids is the x axis. To get the top of each trapezoid connect the y values of each interval.

Graph with 3 trapezoids



Graph with 6 trapezoids



x	$x^2 + 1$
0	1
$\frac{1}{2}$	1.25
1	2
$\frac{3}{2}$	3.25
2	5
$\frac{5}{2}$	7.25
3	10

Area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$

$$A = \frac{1}{2} \left(\frac{1}{2} (1 + 1.25) \right) + \frac{1}{2} \left(\frac{1}{2} (1.25 + 2) \right)$$

The b values in the formula are the parallel bases and h is the height (we can also say the height is the length between each trapezoid)

Calculations:

Area with 3 Trapezoids

$$A = \frac{1}{2}(1) \left(\underbrace{1 + 2 + 2 + 5 + 5 + 10} \right)$$

$$A = \frac{1}{2}(1) (1 + 2(2) + 2(5) + 10)$$

$$A = \frac{25}{2}$$

height of each trapezoid
↓
Area with 6 Trapezoids

$$A = \frac{1}{2} \cdot \frac{1}{2} \left(1 + 2(1.25) + 2(2) + 2(3.25) + 2(5) + 2(7.25) + 10 \right)$$

$$A =$$

If a function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$ then the area under the curve $y = f(x)$ over the interval $[a, b]$ is defined by:

$$Area = \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(x_k) \Delta x$$

Sum up an infinite # rectangles areas
↑ base *↖ height*

Which can be rewritten as :

$$Area = \int_a^b f(x) dx$$

↗ height

Recall the FTC Part I:

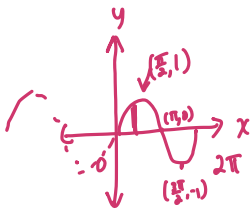
If a function is continuous on $[a, b]$

$$\int_a^b f'(x) dx = f(b) - f(a)$$

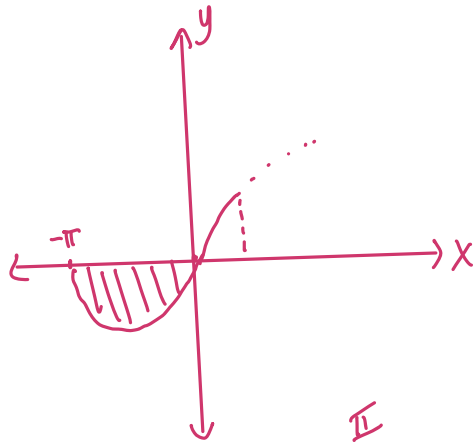
Corollary:

If $f(x) < 0$ on $[a, b]$ then

$$Area = - \int_a^b f(x) dx = \int_b^a f(x) dx$$



Let's find the total area bounded by the curve $y = \sin x$ and the x -axis from $[-\pi, \frac{\pi}{3}]$.



$$\text{Area: } \int_{-\pi}^0 \sin x \, dx + \int_0^{\frac{\pi}{3}} \sin x \, dx = \int_0^{-\pi} \sin x \, dx + \int_0^{\frac{\pi}{3}} \sin x \, dx$$

$$= -\cos x \Big|_0^{-\pi} + -\cos x \Big|_0^{\frac{\pi}{3}}$$

Examples:

1. Find the value of $\int_{-2}^3 x^3 \, dx = \frac{x^4}{4} \Big|_{-2}^3$

$$\frac{3^4}{4} - \frac{(-2)^4}{4} = \frac{81}{4} - \frac{16}{4} = \frac{65}{4}$$

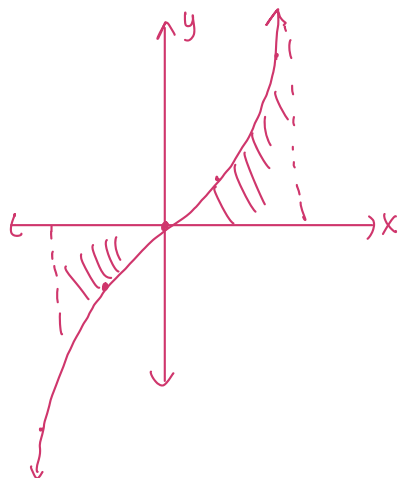
*net area
b/c x^3 is not
always positive/neg
(-2, 3)*

$$= -(\cos(-\pi) - \cos 0) - (\cos \frac{\pi}{3} - \cos 0)$$

$$= -(-1 - 1) - (\frac{1}{2} - 1)$$

$$= 2 - (-\frac{1}{2}) = \frac{5}{2}$$

2. Find the area bounded by $y = x^3$ and the x -axis from $x = -2$ to $x = 3$.



$$A = \int_{-2}^0 x^3 \, dx + \int_0^3 x^3 \, dx$$

$$A = \int_0^{-2} x^3 \, dx + \int_0^3 x^3 \, dx$$

$$= \frac{x^4}{4} \Big|_0^{-2} + \frac{x^4}{4} \Big|_0^3$$

$$4 - 0 + \frac{81}{4} - 0 = 4 + \frac{81}{4} = \frac{16}{4} + \frac{81}{4} = \frac{97}{4}$$

actual area

3. Find the value of $\int_{-2}^2 (x^3 - 4x) dx =$

$\int_{-a}^a \text{odd function} = 0$

$$\left. \frac{x^4}{4} - 2x^2 \right|_{-2}^2$$

$$\left(\frac{2^4}{4} - 2(2)^2 \right) - \left(\frac{(-2)^4}{4} - 2(-2)^2 \right)$$

$$\frac{2^4}{4} - 2(2)^2 - \frac{2^4}{4} + 2(-2)^2 = 0 \rightarrow \text{net area}$$

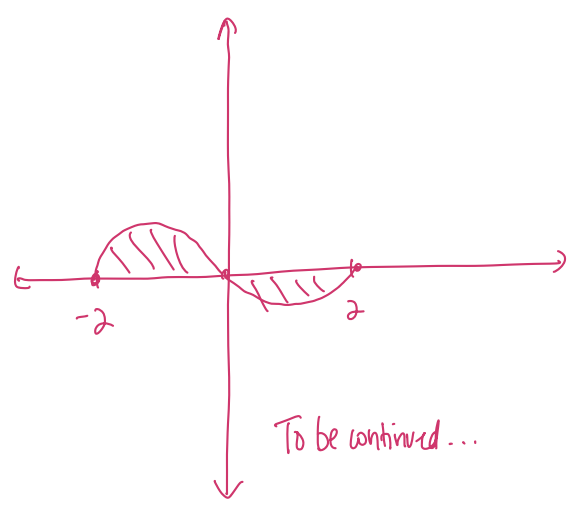
4. Find the area bounded by $y = x^3 - 4x$ and the x-axis from $x = -2$ to $x = 2$.

$$y = x^3 - 4x$$

$$0 = x^3 - 4x$$

$$0 = x(x^2 - 4)$$

$$x = 0, \pm 2$$



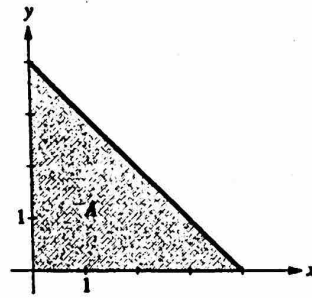
A note about evaluating integrals:

$$\int_{-a}^a \text{odd function} =$$

$$\int_{-a}^a \text{even function} =$$

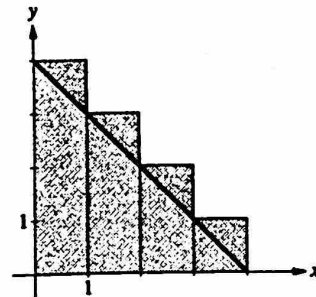
Homework 03-04

1. (a) $\frac{1}{2}bh = \frac{1}{2} \cdot 4 \cdot 4 = 8$



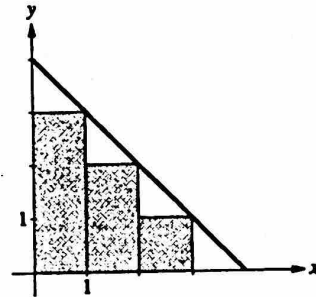
(b) The approximation is greater than A , as the rectangles extend beyond the area.

$$\sum_{k=1}^4 f(x_k^*) \Delta x = (4 + 3 + 2 + 1)(1) = 10$$



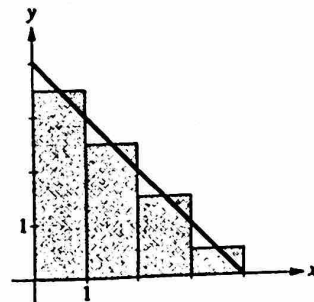
(c) The approximation is less than A , as the rectangles lie inside the area.

$$\sum_{k=1}^4 f(x_k^*) \Delta x = (3 + 2 + 1 + 0)(1) = 6$$



(d) The approximation is equal to A , as can be seen by measuring congruent triangles.

$$\sum_{k=1}^4 f(x_k^*) \Delta x = (3.5 + 2.5 + 1.5 + 0.5)(1) = 8$$



3. (a) $x_k^* = 0, 1, 2, 3, 4$

$$\sum_{k=1}^5 f(x_k^*) \Delta x = (1 + 2 + 5 + 10 + 17)(1) = 35$$

(b) $x_k^* = 1, 2, 3, 4, 5$

$$\sum_{k=1}^5 f(x_k^*) \Delta x = (2 + 5 + 10 + 17 + 26)(1) = 60$$

(c) $x_k^* = 1/2, 3/2, 5/2, 7/2, 9/2$

$$\sum_{k=1}^5 f(x_k^*) \Delta x = (5/4 + 13/4 + 29/4 + 53/4 + 85/4)(1) = 185/4 = 46.25$$

4. (a) $x_k^* = 1, 2, 3, 4, 5$

$$\sum_{k=1}^5 f(x_k^*) \Delta x = (1 + 8 + 27 + 64 + 125)(1) = 225$$

(b) $x_k^* = 2, 3, 4, 5, 6$

$$\sum_{k=1}^5 f(x_k^*) \Delta x = (8 + 27 + 64 + 125 + 216)(1) = 440$$

(c) $x_k^* = 3/2, 5/2, 7/2, 9/2, 11/2$

$$\sum_{k=1}^5 f(x_k^*) \Delta x = (27/8 + 125/8 + 343/8 + 729/8 + 1331/8)(1) = 2555/8 = 319.375$$