Name: ____________________________________ Date: __________

AP Calculus AB: Volumes of Known Cross Sections Ms. Loughran

Do Now:

1. Let $\left(\int_0^x f(t)\,dt\right)' = \left(x\sin\pi x\right)'$, find $f(3)$.

(A) -3π (B) -1 (C) 0 (D) 1 (E) 3π

$f(x) = x\cos(\pi x)\,\pi + \sin \pi x$

$f(3) = 3\cos(3\pi)\cdot\pi + \sin 3\pi = 3(-1)\pi + 0 = -3\pi$

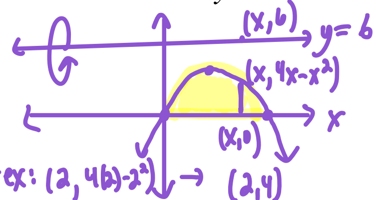
2. Find the volume of the region bounded by $y = 4x - x^2$ and the x-axis when revolved about the line $y = 6$.

$y = 4x - x^2$

$0 = 4x - x^2$

$0 = x(4-x)$

$x = 0, 4$



$R = 6 - 0 = 6$

$r = 6 - 4x + x^2$

$V = \pi\int_0^4 \left(6^2 - (6-4x+x^2)^2\right)dx = \frac{1408\pi}{15}$

A cross section is a slice-not necessarily a disk or a washer.

General solution:

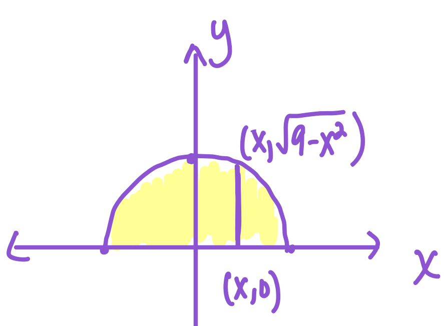
$$V = \int \left(\text{Area of cross section} \perp \text{to the base of the solid}\right) dx \text{ or } dy$$

This formula can be used for solids not obtained by revolution about a line. The only requirement is that each cross section perpendicular to the base of the solid must have a known area.

$y^2 + x^2 = 9$ ← circle

$y^2 = 9 - x^2$ ↙ semicircle

1. The base of solid S is the region enclosed by the graph of $y = \sqrt{9 - x^2}$ and the x-axis. If the cross sections of S perpendicular to the x-axis are squares, find the volume of S.



$A_{square} = s^2$

side of square: $\sqrt{9-x^2} - 0 = \sqrt{9-x^2}$

$V = \int_{-3}^{3} \left(\sqrt{9-x^2}\right)^2 dx = 9x - \frac{x^3}{3}\Big|_{-3}^{3}$

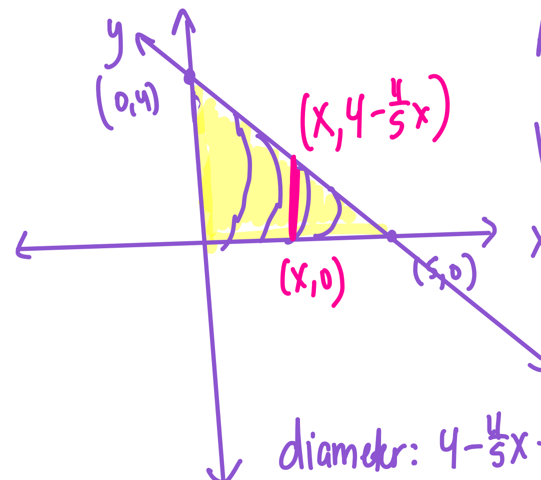
$= 27 - 9 - (-27 + 9) = 18 + 18 = 36$

2. The base of a solid is the region in the first quadrant which is bounded by the line $4x+5y=20$ and the coordinate axes. What is the volume of the solid if every cross section perpendicular to the x-axis is a semicircle?

$4x+5y=20$

$5y=20-4x$

$y=4-\frac{4}{5}x$



$A_{semic} = \frac{1}{2}\pi r^2$

$V=\int_0^5 \frac{1}{2}\pi\left(2-\frac{2}{5}x\right)^2 dx$

$\frac{\pi}{2}\int_0^5 \left(4-\frac{8}{5}x+\frac{4}{25}x^2\right)dx$

$\frac{\pi}{2}\left[4x-\frac{8x^2}{10}+\frac{4x^3}{75}\right]_0^5$

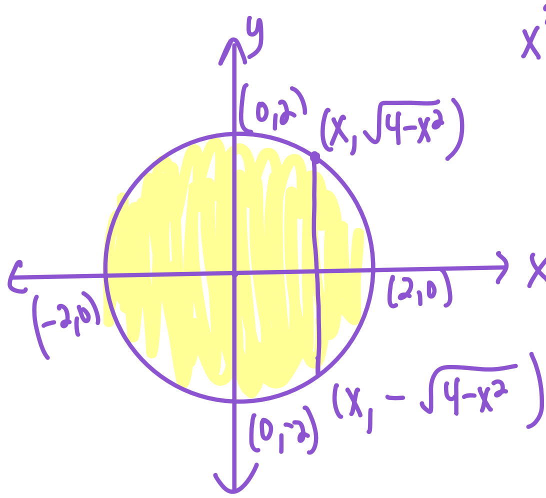
$\frac{\pi}{2}\left[20-\frac{8(5)^2}{10}+\frac{4(5)^3}{75}-0\right]$

diameter: $4-\frac{4}{5}x-0 = 4-\frac{4}{5}x$

radius: $\frac{1}{2}\left(4-\frac{4}{5}x\right) = 2-\frac{2}{5}x$

3. Find the volume of the solid whose base is bounded by the circle $x^2+y^2=4$ with the indicated cross sections taken perpendicular to the x-axis:
(a) squares
(b) equilateral triangles
(c) isosceles right triangles with hypotenuse in bounded region

a)



$x^2+y^2=4$

$y^2=4-x^2$

$y=\pm\sqrt{4-x^2}$

$A_{sq.} = s^2$

$V=\int_{-2}^{2}\left(2\sqrt{4-x^2}\right)^2 dx$

side $=\sqrt{4-x^2}-\left(-\sqrt{4-x^2}\right)$

$=2\sqrt{4-x^2}$

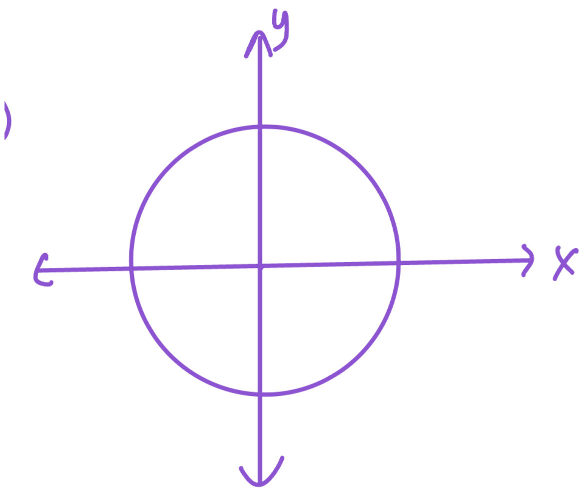
$V=\int_{-2}^{2} 4(4-x^2)\,dx$

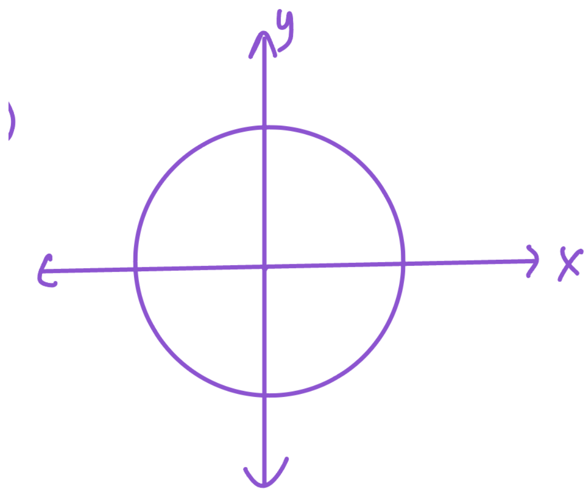
$4\int_{-2}^{2}(4-x^2)dx = 4\left[4x-\frac{x^3}{3}\right]_{-2}^{2}$

$4\left[4(2)-\frac{2^3}{3}-\left(4(-2)-\frac{(-2)^3}{3}\right)\right]$

to be continued ...

Let’s find a general formula for finding the area of an equilateral triangle.





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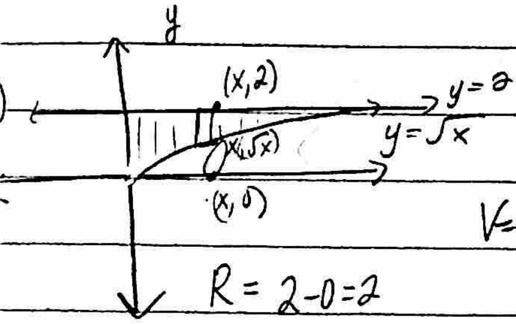
a)



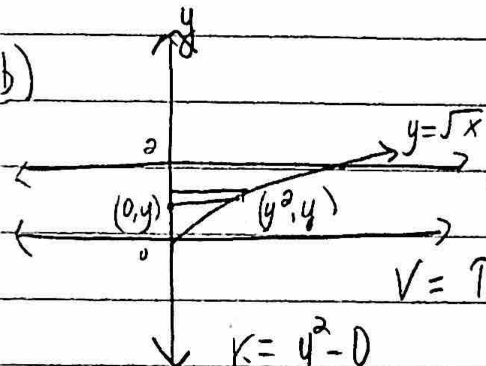
$R = 2-0 = 2$

$r = \sqrt{x} - 0 = \sqrt{x}$

$$V = \pi \int_0^4 \left[(2)^2 - (\sqrt{x})^2\right] dx = \pi \int_0^4 (4-x)\,dx = \pi \cdot \left(4x - \frac{x^2}{2}\right)\Big]_0^4$$

$$\pi\left(4(4) - \frac{4^2}{2} - 0\right) = 8\pi$$

b)



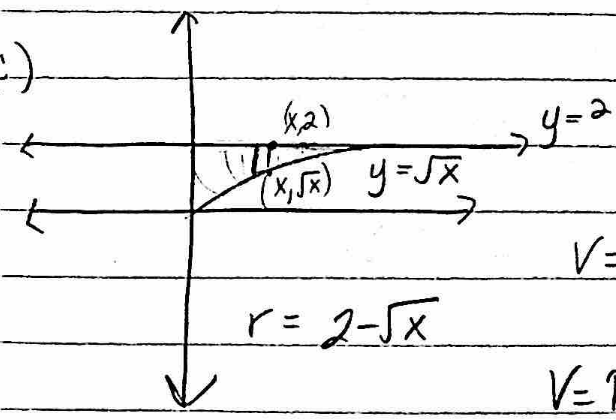
$r = y^2 - 0$

$r = y^2$

$$V = \pi \int_0^2 (y^2)^2\,dy = \pi \int_0^2 y^4\,dy = \pi \cdot \frac{y^5}{5}\Big]_0^2$$

$$\pi\left(\frac{(2)^5}{5}\right) = \frac{32\pi}{5}$$

c)



$r = 2 - \sqrt{x}$

$$V = \pi \int_0^4 (2-\sqrt{x})^2\,dx$$

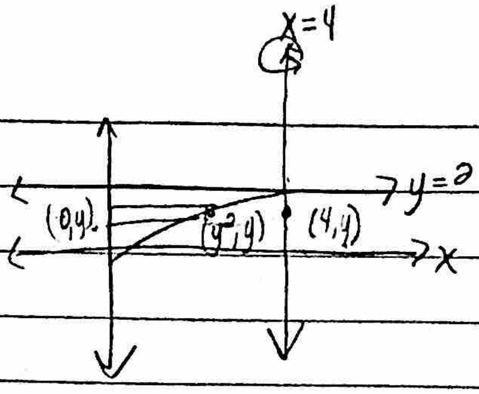
$$V = \pi \int_0^4 (4 - 4\sqrt{x} + x)\,dx$$

$$V = \pi \cdot 4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2}\Big|_0^4$$

$$\pi\left(\left(4(4) - \frac{8(4)^{3/2}}{3} + \frac{4^2}{2}\right) - 0\right) = \pi\left(16 - \frac{64}{3} + 8\right)$$

$$= \pi\left(24 - \frac{64}{3}\right) = \frac{8\pi}{3}$$

(d)



$$V = \pi \int_0^2 \left[(4)^2 - (4-y^2)^2\right] dy$$

$$V = \pi \int_0^2 (16 - (16 - 8y^2 + y^4)) dy$$

$R = 4-0 = 4$

$r = 4 - y^2$

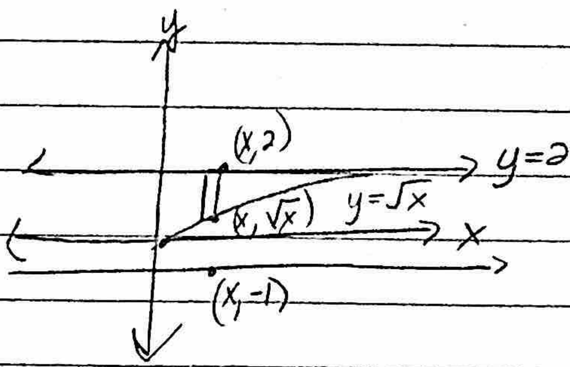
$$= \pi \int_0^2 (8y^2 - y^4) dy$$

$$\pi \cdot \frac{8}{3}y^3 - y^5/5 \Big|_0^2$$

$$\pi\left(\frac{8}{3}(2)^3 - (2)^5/5\right) = \pi\left(\frac{64}{3} - 32/5\right)$$

$$= \frac{224\pi}{15}$$

(e)



$$V = \pi \int_0^4 \left[(3)^2 - (\sqrt{x}+1)^2\right] dx$$

$$V = \pi \int_0^4 9 - (x + 2\sqrt{x} + 1) dx$$

$R = 2-(-1) = 3$

$r = \sqrt{x} - (-1) = \sqrt{x} + 1$

$$V = \pi \int_0^4 (8 - x - 2\sqrt{x}) dx$$

$-\int 2\sqrt{x}$

$-2\int x^{\frac{1}{2}}$

$-2 \cdot \frac{2}{3} x^{3/2}$

$-\frac{4}{3} x^{3/2}$

$$V = \pi \cdot \left(8x - \frac{x^2}{2} - \frac{4x^{3/2}}{3}\right)\Big|_0^4$$

$$V = \pi\left(8(4) - \frac{(4)^2}{2} - \frac{4(4)^{3/2}}{3}\right)$$

$$V = \pi\left(32 - 8 - 32/3\right) = \pi\left(24 - \frac{32}{3}\right)$$

$$= \frac{40\pi}{3}$$