Name: ____________________________ Date: ____________

AP Calc Ms. Loughran

Do Now:

1. Given $f(x) = 3x^2 - 2x + 1$, find $f'(x)$. $= 6x - 2$

$$\lim \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-2x} - 2h \cancel{+1} \cancel{-3x^2} \cancel{+2x} \cancel{-1}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}} = 6x - 2$$

2. Given $g(x) = x^3 + 2x^2 + 3x - 1$, find $g'(x)$. $= 3x^2 + 4x + 3$

$$\lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{+2x^2} + 4xh + 2h^2 \cancel{+3x} + 3h \cancel{-1} \cancel{-x^3} \cancel{-2x^2} \cancel{-3x} \cancel{+1}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 + 4x + 2h + 3)}{\cancel{h}} = 3x^2 + 4x + 3$$

3. Given $f(x) = 2x$, find $f'(x)$. $= 2$

$$\lim_{h \to 0} \frac{\cancel{2x} + 2h \cancel{-2x}}{h} = \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}} = \lim_{h \to 0} 2 = 2$$

4. Given $f(x) = 5$, find $f'(x)$. $= 0$

$$\lim_{h \to 0} \frac{5 - 5}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

* the derivative of a constant is always 0.

From our work over the past few days, we have found that:

if $f(x) = x^2 + 4x + 9$, then $f'(x) = 2x + 4$

if $f(x) = x^2 - 5x + 1$, then $f'(x) = 2x - 5$

if $f(x) = 3x^2 - 4$, then $f'(x) = 6x$

if $f(x) = 2x^2 - 5x^3$, then $f'(x) = 4x - 15x^2$

if $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$

$f(x) = x^{-1}$ $\quad$ $f'(x) = -x^{-2}$

Do you see a shortcut for finding $f'(x)$? (Make sure your assumption works for all the previous examples above.)

Name:________________________________ Date:__________

AP Calc AB: Derivatives Ms. Loughran

Notation

There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief, but does not name the independent variable.
$\dfrac{dy}{dx}$	"dy dx" or "the derivative of y with respect to x"	Names both variables and uses d for derivative.
$\dfrac{df}{dx}$	"df dx" or "the derivative of f with respect to x"	Emphasizes the function's name.
$\dfrac{d}{dx}f(x)$	"d dx of f at x" or "the derivative of f at x"	Emphasizes the idea that differentiation is an operation performed on f.

RULE 1 Derivative of a Constant Function

If f is the function with the constant value c, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

RULE 2 Power Rule for Positive Integer Powers of x

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

RULE 3 The Constant Multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

RULE 4 The Sum and Difference Rule

If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}.$$

In function notation,

$$(f + g)' = f' + g' \qquad (f - g)' = f' - g'$$

Examples:

1. Find $\frac{dp}{dt}$ if $p = t^3 + 6t^2 - \frac{5}{3}t + 16$

$$\frac{dp}{dt} = 3t^2 + 12t - \frac{5}{3}$$

2. Find $\frac{dy}{dx}$ if $y = -x^2 + 3$

$$\frac{dy}{dx} = -2x$$

3. Find $\frac{dy}{dx}$ if $y = \frac{x^3}{3} - 3$ $= \frac{1}{3}x^3 - 3$

$$\frac{dy}{dx} = x^2$$

4. Find $\frac{dy}{dx}$ if $y = x^2 + x + 1$

$$\frac{dy}{dx} = 2x + 1$$

5. Find $\frac{dy}{dx}$ if $y = x^4 - 7x^3 + 2x^2 + 15$

$$\frac{dy}{dx} = 4x^3 - 21x^2 + 4x$$

6. Find $\frac{dy}{dx}$ if $y = 4x^{-2} - 8x + 1$

$$\frac{dy}{dx} = -8x^{-3} - 8 = \frac{-8}{x^3} - 8$$

7. Find $\frac{dy}{dx}$ if $y = \frac{x^{-4}}{4} - \frac{x^{-3}}{3} + \frac{x^{-2}}{2} - x^{-1} + 3$

$$\frac{dy}{dx} = -x^{-5} + x^{-4} - x^{-3} + x^{-2} = -\frac{1}{x^5} + \frac{1}{x^4} - \frac{1}{x^3} + \frac{1}{x^2}$$

8. Use the definition of derivative to show that $\frac{d}{dx}(x) = 1$.

$f(x) = x$

"the derivative of x is 1"

$$\lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

Homework 09-26

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

4. Given $f(x) = 4x^3 - 5x$
   (a) Find $f'(x)$. $= 12x^2 - 5$
   (b) Find $f'(-1)$, $f'(0)$, and $f'(2)$.
   (c) Write an equation of the tangent line to $f(x)$ at $x = -1$.

b) $f'(-1) = 7$

$f'(0) = -5$

$f'(2) = 43$

c) $f(-1) = 1$

$y - 1 = 7(x+1)$

5. Given $f(x) = \sqrt{8x}$
   (a) Find $f'(x)$.
   (b) Find $f'(2)$.
   (c) Write an equation of the tangent line of $f(x)$ at $x = 2$.

a) $f'(x) = \dfrac{4}{\sqrt{8x}}$ or $\dfrac{8}{2\sqrt{8x}}$

b) $f'(2) = 1$

c) $f(2) = 4$

$y - 4 = 1(x-2)$

Quick Review 3.1 (For help, go to Sections 2.1 and 2.4.)

In Exercises 1–4, evaluate the indicated limit algebraically.

1. $\lim_{h\to 0} \frac{(2+h)^2-4}{h}$

2. $\lim_{x\to 2^+} \frac{x+3}{2}$

3. $\lim_{y\to 0^-} \frac{|y|}{y}$

4. $\lim_{x\to 4} \frac{2x-8}{\sqrt{x}-2}$

5. Find the slope of the line tangent to the parabola $y = x^2 + 1$ at its vertex.

6. By considering the graph of $f(x) = x^3 - 3x^2 + 2$, find the intervals on which f is increasing.

In Exercises 7–10, let

$$f(x) = \begin{cases} x+2, & x \le 1 \\ (x-1)^2, & x > 1. \end{cases}$$

7. Find $\lim_{x\to 1^+} f(x)$ 0 and $\lim_{x\to 1^-} f(x)$. 3

8. Find $\boxed{\lim_{h\to 0^+} f(1+h)}$. $= f(1)$

9. Does $\lim_{x\to 1} f(x)$ exist? Explain. No $\lim_{x\to 1^-} f(x) \ne \lim_{x\to 1^+} f(x)$

10. Is f continuous? Explain. f is not continuous at 1 bc $\lim_{x\to 1} f(x)$ dne

$\lim_{h\to 0} f(1+h) = f(1+0) = f(1) = 1+2 = 3$

① $\lim_{h\to 0} \frac{((2+h)-2)(2+h+2)}{h}$

$\lim_{h\to 0} \frac{\cancel{h}(h+4)}{\cancel{h}} = 4$

② $\lim_{x\to 2^+} \frac{x+3}{2} = \frac{2+3}{2} = \frac{5}{2}$

③ $\lim_{y\to 0^-} \frac{|y|}{y} = -1$

④ $\lim_{x\to 4} \frac{2x-8}{\sqrt{x}-2}$ $\quad 2(x-4)$

$\lim_{x\to 4} \frac{2\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)}{\cancel{\sqrt{x}-2}} = 2(4) = 8$

Quick Review 3.1 (For help, go to Sections 2.1 and 2.4.)

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, evaluate the indicated limit algebraically.

1. $\lim_{h\to 0} \dfrac{(2+h)^2-4}{h}$
2. $\lim_{x\to 2^+} \dfrac{x+3}{2}$
3. $\lim_{y\to 0^-} \dfrac{|y|}{y}$
4. $\lim_{x\to 4} \dfrac{2x-8}{\sqrt{x}-2}$
5. Find the slope of the line tangent to the parabola $y = x^2 + 1$ at its vertex.
6. By considering the graph of $f(x) = x^3 - 3x^2 + 2$, find the intervals on which f is increasing.

In Exercises 7–10, let

$$f(x) = \begin{cases} x+2, & x \le 1 \\ (x-1)^2, & x > 1. \end{cases}$$

7. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.
8. Find $\lim_{h\to 0^+} f(1+h)$.
9. Does $\lim_{x\to 1} f(x)$ exist? Explain.
10. Is f continuous? Explain.

Section 3.1 Exercises

In Exercises 1–4, use the definition

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

to find the derivative of the given function at the given value of a.

1. $f(x) = 1/x,\ a = 2$
2. $f(x) = x^2 + 4,\ a = 1$
3. $f(x) = 3 - x^2,\ a = -1$
4. $f(x) = x^3 + x,\ a = 0$

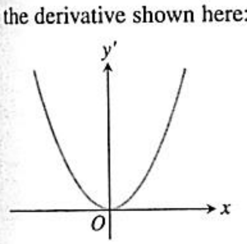
In Exercises 5–8, use the definition

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

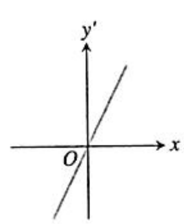
to find the derivative of the given function at the given value of a.

5. $f(x) = 1/x,\ a = 2$
6. $f(x) = x^2 + 4,\ a = 1$
7. $f(x) = \sqrt{x+1},\ a = 3$
8. $f(x) = 2x + 3,\ a = -1$
9. Find $f'(x)$ if $f(x) = 3x - 12$.
10. Find dy/dx if $y = 7x$.
11. Find $\dfrac{d}{dx}(x^2)$.
12. Find $\dfrac{d}{dx} f(x)$ if $f(x) = 3x^2$.

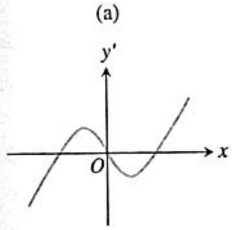
In Exercises 13–16, match the graph of the function with the graph of the derivative shown here:



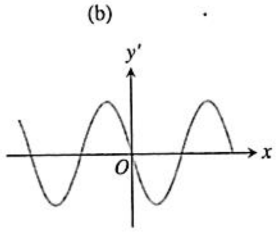
(a)



(b)

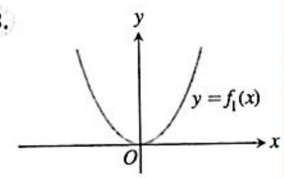


(c)

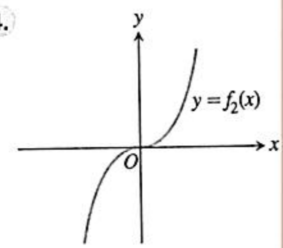


(d)

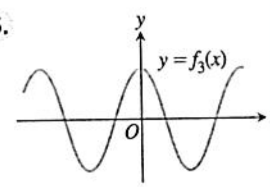
13.



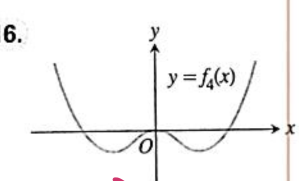
14.



15.



16.



① $-\frac{1}{4}$

② 2

③ 2

④ 1

⑰ a) $y-3=5(x-2)$

b) $y-3=-\frac{1}{5}(x-2)$

⑱ $y' = 4x-13$

tan line eq: $y = -x-13$

⑲ a) $y=3x-2$

b) $y=-\frac{1}{3}x+\frac{4}{3}$

⑳ a) $y-2=\frac{1}{4}(x-4)$

b) $y-2=-4(x-4)$

(2,3)

17. If $f(2) = 3$ and $f'(2) = 5$, find an equation of (a) the *tangent* line, and (b) the *normal* line to the graph of $y = f(x)$ at the point where $x = 2$.
[*Hint:* Recall that the normal line is perpendicular to the tangent line.]

slope of tan line at $x=2$

a) $y-3=5(x-2)$

b) $y-3=-\frac{1}{5}(x-2)$

19. Find the lines that are **(a)** tangent and **(b)** normal to the curve $y = x^3$ at the point $(1, 1)$.

$$(x+h)^3 = (x+h)(x^2+2xh+h^2)$$
$$= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3$$
$$= x^3 + 3x^2h + 3xh^2 + h^3$$

$$y' = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} = \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} = 3x^2$$

$$y'(1) = 3(1)^2 = 3$$

a) $y - 1 = 3(x-1)$

b) $y - 1 = -\frac{1}{3}(x-1)$

4. $f(x) = x^3 + x, a = 0$

$$\lim_{h\to 0} \frac{(x+h)^3 + (x+h) - (x^3+x)}{h}$$

$$\lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{x} + h - \cancel{x^3} - \cancel{x}}{h}$$

$$\lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 + 1)}{\cancel{h}} = 3x^2 + 1$$

shortcut $y = x^{\frac{1}{2}}$
$y' = \frac{1}{2}x^{-\frac{1}{2}}$

20 $y = \sqrt{x}$

$$\lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$\cancel{x} + h - \cancel{x}$

$$\lim_{h\to 0} \frac{\cancel{h}\,1}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

$y(4) = \sqrt{4} = 2$
$(4, 2)$

$y'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

TL: $y - 2 = \frac{1}{4}(x-4)$

NL: $y - 2 = -4(x-4)$

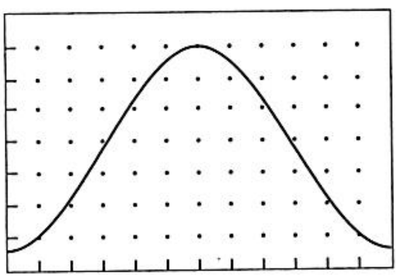
18. Find the derivative of the function $y = 2x^2 - 13x + 5$ and use it to find an equation of the line tangent to the curve at $x = 3$.

19. Find the lines that are **(a)** tangent and **(b)** normal to the curve $y = x^3$ at the point $(1, 1)$.

20. Find the lines that are (a) tangent and (b) normal to the curve $y = \sqrt{x}$ at $x = 4$.

$y = x^{\frac{1}{2}}$ $\quad y' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

21. ***Daylight in Fairbanks*** The viewing window below shows the number of hours of daylight in Fairbanks, Alaska, on each day for a typical 365-day period from January 1 to December 31. Answer the following questions by estimating slopes on the graph in hours per day. For the purposes of estimation, assume that each month has 30 days.



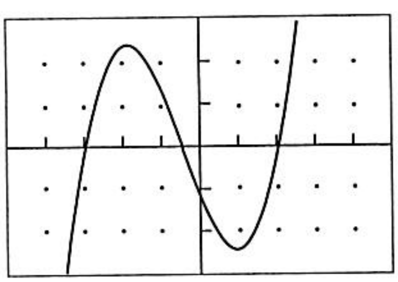
[0, 365] by [0, 24]

(a) On about what date is the amount of daylight increasing at the fastest rate? What is that rate?

(b) Do there appear to be days on which the rate of change in the amount of daylight is zero? If so, which ones?

(c) On what dates is the rate of change in the number of daylight hours positive? negative?

22. ***Graphing f′ from f*** Given the graph of the function f below, sketch a graph of the *derivative* of f.



[–5, 5] by [–3, 3]

23. The graphs in Figure 3.10a show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on the rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 3.10b shows the graph of the derivative of the rabbit population. We made it by plotting slopes, as in Example 3.

(a) What is the value of the derivative of the rabbit population in Figure 3.10 when the number of rabbits is largest? smallest?

(b) What is the size of the rabbit population in Figure 3.10 when its derivative is largest? smallest?

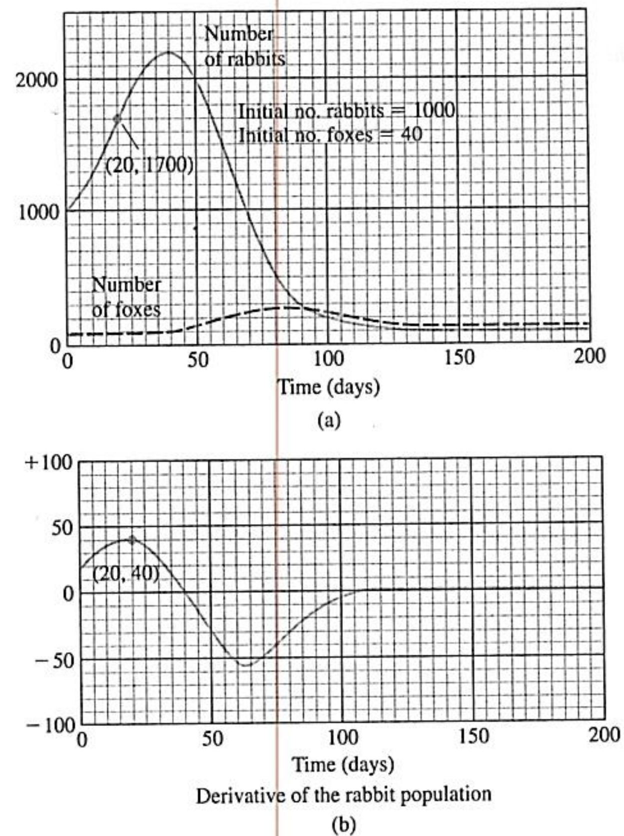
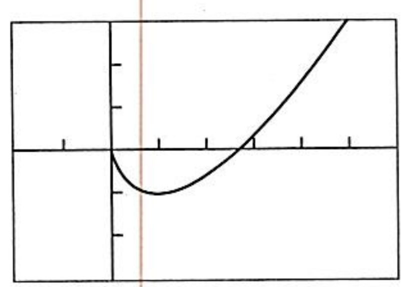


Figure 3.10 Rabbits and foxes in an arctic predator-prey food chain. *Source: Differentiation* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, MA, 1975, p. 86.

24. Shown below is the graph of $f(x) = x \ln x - x$. From what you know about the graphs of functions (i) through (v), pick out the one that is the *derivative* of f for $x > 0$.



[–2, 6] by [–3, 3]

i. $y = \sin x$ **ii.** $y = \ln x$ **iii.** $y = \sqrt{x}$
iv. $y = x^2$ **v.** $y = 3x - 1$

25. From what you know about the graphs of functions (i) through (v), pick out the one that is *its own derivative.*

i. $y = \sin x$ **ii.** $y = x$ **iii.** $y = \sqrt{x}$
iv. $y = e^x$ **v.** $y = x^2$

① $f(x) = \frac{1}{x}$ $\quad f'(2) =$

$$\lim_{h \to 0} \frac{\frac{1}{\cancel{x+h}}^{\,x(\cancel{x+h})} - \frac{1}{\cancel{x}}^{\,\cancel{x}(x+h)}}{h\,x(x+h)}$$

$$\lim_{h \to 0} \frac{\overset{-h}{\cancel{x} - (\cancel{x}+h)}}{xh(x+h)}$$

$$\lim_{h \to 0} \frac{-\cancel{h}^{\,1}}{x\cancel{h}(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

$$f'(2) = -\frac{1}{(2)^2} = -\frac{1}{4}$$