Name: ________________________________ Date: ________

AP Calc AB: Derivatives of $a^x$ and $\log_a x$ Ms. Loughran

$f(x)=\begin{cases} \ln x & x>0 \\ \ln(-x) & x<0 \end{cases}$ $f'(x)=\begin{cases} \frac{1}{x} & x>0 \\ \frac{1}{-x}\cdot -1 = \frac{1}{x} & x<0 \end{cases}$

Do Now:

1. If $f(x)=\ln|x|$, find $f'(x)$. $=\frac{1}{x}$

**Derivative of $a^x$**

If $a>0$ and $a\neq 1$ we can use the properties of logarithms to write $a^x$ in terms of $e^x$:

$$a^x = e^{\ln a^x} = e^{x\ln a}$$

We can then find the derivative of $a^x$ with the Chain Rule

constant

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot \ln a$$

$$a^x \cdot \ln a$$

$$\boxed{\frac{d}{dx}(a^u) = a^u \cdot \ln a \cdot u'}$$

**Derivative of $\log_a x$**

To find the derivative of $\log_a x$ for an arbitrary base $(a>0,\ a\neq 1)$ we use the change of base formula for logarithms to express $\log_a x$ in terms of natural logarithms:

* change of base formula

$y=\log_a x$

$a^y = x$

$\ln a^y = \ln x$

$y\ln a = \ln x$

$y=\frac{\ln x}{\ln a}$

constant

$$\log_a x = \frac{\ln x}{\ln a} = \frac{1}{\ln a}\cdot \ln x \qquad x>0$$

$$\frac{d}{dx}\left[\frac{1}{\ln a}\cdot \ln x\right] = \frac{1}{\ln a}\cdot\frac{1}{x} = \frac{1}{x\ln a}$$

$$\boxed{\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\cdot u'}$$

For each of the following find $\dfrac{dy}{dx}$.

1. $y = 4^x$

$\frac{dy}{dx} = 4^x \cdot \ln 4 \cdot 1 = 4^x \ln 4$

* you can not take the log of 0 or a negative #

$x > 0$

2. $y = \log_5 \sqrt{x}$

$\frac{dy}{dx} = \frac{1}{\sqrt{x}\ln 5} \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}\ln 5} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x\ln 5}$

3. $y = 3^{6x}$

$\frac{dy}{dx} = 3^{6x} \ln 3 \cdot 6$

4. $y = 5^{\cos x}$

$\frac{dy}{dx} = 5^{\cos x} \ln 5 (-\sin x)$

5. $y = \log_3(\underset{u}{1 + x\ln 3})$

$\frac{dy}{dx} = \frac{1}{(1+x\ln 3)\cancel{\ln 3}} \cdot \cancel{\ln 3} = \frac{1}{1+x\ln 3}$

$1 + x\ln 3 > 0$

$x\ln 3 > -1$

$x > -\frac{1}{\ln 3}$

6. $y = 2^{\sin x}$

$\frac{dy}{dx} = 2^{\sin x} \cdot \ln 2 \cdot \cos x$

7. $y = \ln 2 \cdot \log_2 x$

$\frac{dy}{dx} = \cancel{\ln 2} \cdot \frac{1}{x\cancel{\ln 2}} = \frac{1}{x}$ $\quad x > 0$

8. $y = \log_{10} e^x$

$\frac{dy}{dx} = \frac{1}{\cancel{e^x}\ln 10} \cdot \cancel{e^x} = \frac{1}{\ln 10}$

you dont need a restriction b/c $e^x$ is always $\oplus$

9. $y = 3^{\cot x}$

$\frac{dy}{dx} = 3^{\cot x} \cdot \ln 3 \cdot (-\csc^2 x)$

10. $y = 9^{-x}$

$\frac{dy}{dx} = 9^{-x} \cdot \ln 9 \cdot -1$

11. $y = e^{x^2}$

$\frac{dy}{dx} = e^{x^2} \cdot \ln e \cdot 2x = e^{x^2} \cdot 2x$ $\quad \underline{\text{or}} \quad [e^u]' = e^{x^2} \cdot 2x$

12. $y = \log_a a^{\sin x}$

$\frac{dy}{dx} = \frac{1}{\cancel{a^{\sin x}}\ln a} \cdot \cancel{a^{\sin x}\ln a} \cdot \cos x = \cos x$

you could rewrite $\quad \underline{\text{OR}}$

$\log_a a^{\sin x} = \sin x$

$\frac{d}{dx} \sin x = \cos x$

# Homework 11-01

This is the key to the entire packet.

## Implicit Differentiation Packet

1) $x^2+y^2=5$

$2x+2y\,dy/dx=0$

$2y\,dy/dx=-2x$

$dy/dx=-2x/2y=-x/y$

2) $x^2-xy+y^3=1$

$2x-(x\,dy/dx+1\cdot y)+3y^2\,dy/dx=0$

$2x-x\,dy/dx-y+3y^2\,dy/dx=0$

$-x\,dy/dx+3y^2\,dy/dx=-2x+y$

$dy/dx(-x+3y^2)=-2x+y$

$dy/dx=\dfrac{-2x+y}{-x+3y^2}$

(b) $(1,0)$ $dy/dx=\dfrac{-2(1)+0}{-1+3(0)^2}=\dfrac{-2}{-1}=2$

$(1,1)$ $dy/dx=\dfrac{-2(1)+1}{-1+3(1)^2}=\dfrac{-1}{2}$

$(1,-1)$ $dy/dx=\dfrac{-2(1)+(-1)}{-1+3(-1)^2}=\dfrac{-3}{2}$

when $x=1$ $1^2-(1)y+y^3=1$

$y^3-y=0$

$y(y^2-1)=0$

$y=0 \mid y=\pm1$

3) $x^2+y^2-12x-6y+25=0$

$2x+2y\,dy/dx-12-6\,dy/dx=0$

$dy/dx(2y-6)=-2x+12$

$dy/dx=\dfrac{-2x+12}{2y-6}=\dfrac{\cancel{2}(-x+6)}{\cancel{2}(y-3)}$

$x^2+y^2+2x+y-10=0$

$2x+2y\,dy/dx+2+dy/dx=0$

$dy/dx(2y+1)=-2x-2$

$dy/dx=\dfrac{-2x-2}{2y+1}=$

Unless a point is stated as being on a curve do not assume it is. Neither curve goes through (0,0)

$\dfrac{-x+6}{y-3}=\dfrac{-2x-2}{2y+1}$

$(-x+6)(2y+1)=(y-3)(-2x-2)$

$\dfrac{-x+6}{-2x-2}=\dfrac{y-3}{2y+1}$ $\dfrac{-2}{3}$

$-\frac{2}{3}$ $(0,0)$ ✓ not on curves

$(2,1)$ ✓ (A)

4) $(x+y)^2 = (2x-y)^3$ at $(0,-1)$

$2(x+y)\cdot(1+dy/dx) = 3(2x-y)^2\cdot(2-1\,dy/dx)$

sub in $(0,-1)$

$2(0-1)\cdot(1+dy/dx) = 3(2(0)-(-1))^2\cdot(2-1\,dy/dx)$

$-2(1+dy/dx) = 3(2-1\,dy/dx)$

$-2-2\,dy/dx = +6-3\,dy/dx$

$-8 = -3\,dy/dx + 2\,dy/dx$

$-8 = -dy/dx$

$8 = dy/dx$

5) $y = (3+6x+\sqrt{x^2+9})^{1/5}$

$y^5 = 3+6x+\sqrt{x^2+9}$

$y^5 = 3+6x+(x^2+9)^{1/2}$

$5y^4\,dy/dx = 0+6+\frac{1}{2}(x^2+9)^{-1/2}\cdot(2x)$

$5y^4\,dy/dx = 6+\frac{x}{\sqrt{x^2+9}}$

$\frac{dy}{dx} = \frac{6+\frac{x}{\sqrt{x^2+9}}}{5y^4}$

$\frac{dy}{dx} = \frac{6\sqrt{x^2+9}+x}{5y^4(\sqrt{x^2+9})}$

when $x=4$

$y=(3+6(4)+\sqrt{4^2+9})^{1/5}$

$y=(3+24+5)^{1/5}$

$y=32^{1/5}=2$

$\left.\frac{dy}{dx}\right|_{\substack{x=4\\y=2}} \quad \frac{6\sqrt{4^2+9}+4}{5(2)^4(\sqrt{4^2+9})} = \frac{34}{400} = \frac{17}{200}$

⑥ $x^2-y^2=1$

(a) $2x-2y\,{}^{dy}\!/_{dx}=0$

$-2y\,{}^{dy}\!/_{dx}=-2x$

${}^{dy}\!/_{dx}=\frac{-2x}{-2y}=\frac{x}{y}$

(b) $\left({}^{dy}\!/_{dx}\right)'=\left(\frac{x}{y}\right)'=\frac{y\cdot 1-x\cdot {}^{dy}\!/_{dx}}{y^2}=\frac{y-x\,{}^{dy}\!/_{dx}}{y^2}=\frac{y-x\left(\frac{x}{y}\right)}{y^2}=\frac{y-\frac{x^2}{y}}{y^2}$

$\frac{y^2-x^2}{y^3}$

$=\frac{-1(-y^2+x^2)}{y^3}=\frac{-1(1}{y^3}$

$=-{}^{1}\!/_{y^3}$

(c) $\left({}^{d^2y}\!/_{dx^2}\right)'=\left(-\frac{1}{y^3}\right)'=-1\left(\frac{1}{y^3}\right)'=-1\cdot\frac{-3y^2\,{}^{dy}\!/_{dx}}{(y^3)^2}$

$=-1\cdot\frac{-3y^2\cdot\frac{x}{y}}{y^6}$

$=-1\cdot\frac{-3xy}{y^6}=\frac{3xy}{y^6}=\frac{3x}{y^5}$ ✓

⑦ I. $y^4=x^2+5$

$4y^3\,{}^{dy}\!/_{dx}=2x$

${}^{dy}\!/_{dx}={}^{2x}\!/_{4y^3}=\frac{x}{2y^3}$ (D)

II. $x^2=a^2+y^2$

$2x=0+2y\,{}^{dy}\!/_{dx}$

$\frac{2x}{2y}=\frac{dy}{dx}$ (E)

$\frac{x}{y}={}^{dy}\!/_{dx}$

III. $x = -y - xy$

$1 = -1\,dy/dx - (x\,dy/dx + 1 \cdot y)$

$1 = -dy/dx - x\,dy/dx - y$

$1 + y = -dy/dx - x\,dy/dx$

$1 + y = dy/dx(-1 - x)$

$\frac{1+y}{-1-x} = dy/dx$ (G)

IV $y^2 + xy = 3$

$2y\,dy/dx + x\,dy/dx + 1y = 0$

$dy/dx\,(2y + x) = -y$

$dy/dx = -y/2y+x$ (B)

V $x^2 = a^2 - y^2$

$2x = -2y\,dy/dx$

$\frac{2x}{-2y} = dy/dx$

$-x/y = dy/dx$ (C)

VI $x^3 + y^3 - a^3 = 0$

$3x^2 + 3y^2\,dy/dx = 0$

$dy/dx = -3x^2/3y^2 = -x^2/y^2$ (A)

VII $x - y = xy$

$1 - dy/dx = x\,dy/dx + y$

$1 - y = x\,dy/dx + dy/dx$

$1 - y = dy/dx\,(x+1)$

$\frac{1-y}{x+1} = \frac{dy}{dx}$ (F)

* when x and y are near 0
$y^5$ approaches 0 much quicker than $x^3$ or $y^3$
So...
$$y^5+y^3=x^3$$
$$y^3=x^3$$
$$y=x \text{ so } y'=1$$

8) $y^5+y^3=x^3$

$$5y^4\,dy/dx+3y^2\,dy/dx=3x^2$$
$$dy/dx(5y^4+3y^2)=3x^2$$
$$\frac{dy}{dx}=\frac{3x^2}{5y^4+3y^2}\Bigg|_{(0,0)} \quad \frac{3(0)^2}{5(0)^4+3(0)^2} \text{ indeterminate}$$

9) $y^3+3x^2y+13=0$

(a) $3y^2\,dy/dx+3x^2\,dy/dx+6x\cdot y=0$
$$3y^2\,dy/dx+3x^2\,dy/dx=-6xy$$
$$dy/dx(3y^2+3x^2)=-6xy$$
$$dy/dx=\frac{-6xy}{3y^2+3x^2}=\frac{-\cancel{6}^{2}xy}{\cancel{3}(y^2+x^2)}=\frac{-2xy}{y^2+x^2}$$

(b) $dy/dx\Big|_{(2,-1)} \quad \frac{-2(2)(-1)}{(-1)^2+(2)^2}=\frac{4}{5}$

$$y+1=4/5(x-2)$$

10) $x^2-xy+y^2=9$

) $2x-(x\,dy/dx+1y)+2y\,dy/dx=0$
$$2x-x\,dy/dx-y+2y\,dy/dx=0$$
$$2x-y=x\,dy/dx-2y\,dy/dx$$
$$2x-y=dy/dx(x-2y)$$
$$\frac{2x-y}{x-2y}=\frac{dy}{dx}$$

(b) Tangents are vertical when slope is undefined.

$$x-2y=0$$
$$-2y=-x$$
$$2y=x$$

once you have y then plug into $2y=x$

$$(2y)^2-(2y)(y)+y^2=9$$
$$4y^2-2y^2+y^2=9$$
$$3y^2=9$$
$$y^2=3$$
$$y=\pm\sqrt{3}$$

put into original problem to get other variable

$$2(\sqrt{3})=2\sqrt{3}$$
$$2(-\sqrt{3})=-2\sqrt{3}$$

$(2\sqrt{3},\sqrt{3})$

$(-2\sqrt{3},-\sqrt{3})$

(c) $dy/dx\Big|_{(0,3)}$ $\dfrac{2(0)-3}{0-2(3)}=\dfrac{-3}{-6}=\dfrac{1}{2}$

$$\frac{dy}{dx}=\frac{y-2x}{2y-x}$$

$$d^2y/dx^2=\frac{(2y-x)(dy/dx-2)-(y-2x)(2\,dy/dx-1)}{(2y-x)^2}$$

rate of change → slope

rate of change of slope → 2nd derivative

$$=\frac{(2(3)-0)(\tfrac{1}{2}-2)-(3-2(0))(2(\tfrac{1}{2})-1)}{(2(3)-0)^2}$$

$$=\frac{6(-1.5)-0}{36}=\frac{-9}{36}=\frac{-1}{4}$$

Don't worry about clean up just plug values in.

11 $xy^2 - x^3y = 6$

(a)

$x \cdot 2y\, dy/dx + 1y^2 - (x^3\, dy/dx + 3x^2y) = 0$

$2xy\, dy/dx + y^2 - x^3\, dy/dx - 3x^2y = 0$

$2xy\, dy/dx - x^3\, dy/dx = -y^2 + 3x^2y$

$dy/dx(2xy - x^3) = -y^2 + 3x^2y$

$$\frac{dy}{dx} = \frac{-y^2 + 3x^2y}{2xy - x^3}$$

(b) $xy^2 - x^3y = 6$

$(1)y^2 - (1)^3y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3 \mid y = -2$

$(1,3)$

$(1,-2)$

$$dy/dx\Big|_{(1,3)} \frac{\overset{-9}{-(3)^2} + 3(1)^2(3)}{2(1)(3) - 1^3} = \frac{0}{5} = 0$$

$y = 3$

$$dy/dx\Big|_{(1,-2)} \frac{-(-2)^2 + \overset{-6}{3(1)^2(-2)}}{\underset{-4}{2(1)(-2)} - \underset{-1}{(1)^3}} = \frac{-10}{-5} = 2$$

$y + 2 = 2(x-1)$

(c) Tangent line is vertical when m is undefined.

$2xy - x^3 = 0$

$2xy = x^3$

$x(2y - x^2) = 0$

$x = 0$ or $y = \frac{1}{2}x^2$

$x \neq 0$ b/c of given eq. $xy^2 - x^3y = 6$

$0(y^2) - 0 \cdot y = 6$

$0 \neq 6$

$x\left(\frac{1}{2}x^2\right)^2 - x^3\left(\frac{1}{2}x^2\right) = 6$

$\frac{1}{4}x^5 - \frac{1}{2}x^5 = 6$

$-\frac{1}{4}x^5 = 6$

$-x^5 = \cdot 24$

$x = \sqrt[5]{-24}$

(12) $x + xy + 2y^2 = 6$

(a) $1 + x\,{}^{dy}/_{dx} + 1y + 4y\,{}^{dy}/_{dx} = 0$

$$x\,{}^{dy}/_{dx} + 4y\,{}^{dy}/_{dx} = -1 - y$$
$${}^{dy}/_{dx}(x + 4y) = -1 - y$$
$$\frac{dy}{dx} = \frac{-1-y}{x+4y}$$

(b) $\left.\frac{dy}{dx}\right|_{(2,1)} \quad \frac{-1-1}{2+4(1)} = \frac{-2}{6} = \frac{-1}{3}$

$$y - 1 = -{}^{1}/_{3}(x - 2)$$

(c) $\frac{-1-y}{x+4y} = \frac{-1}{3}$

$$-3 - 3y = -x - 4y$$
$$3 + 3y = x + 4y$$
$$3 - x = y$$ put into original eq.

$9 - 6x + x^2$

$$x + x(3 - x) + 2(3 - x)^2 = 6$$
$$x + 3x - x^2 + 18 - 12x + 2x^2 = 6$$
$$x^2 - 8x + 12 = 0$$
$$(x - 6)(x - 2) = 0$$

$x = 6 \mid x = 2$

Remember $3 - x = y$

If $x = 6$

$y = 3 - 6 = -3$

$(6, -3)$

If $x = 2$

$y = 3 - 2 = 1$

$(2, 1)$