Name: ________________________________ Date: ___________

AP Calculus AB – Introduction to Related Rates Ms. Loughran

1. Given $y = 4x^3 - 3x^2 - 9x + 1$, find $\frac{dy}{dx} = 12x^2 - 6x - 9$

2. Given $y = 4w^3 - 3w^2 - 9w + 1$, find $\frac{dy}{dw} = 12w^2 - 6w - 9$

3. Given $v = 4m^3 - 3m^2 - 9m + 1$, find $\frac{dv}{dm} = 12m^2 - 6m - 9$

4. Given $y = 4x^3 - 3x^2 - 9x + 1$, find $\frac{dy}{dt} = 12x^2 \frac{dx}{dt} - 6x \frac{dx}{dt} - 9 \frac{dx}{dt}$

5. Given $P = 2L + 2W$, find $\frac{dP}{dt} = 2 \frac{dL}{dt} + 2 \frac{dW}{dt}$

6. Given $A = \pi r^2$, find $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

7. Given $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Volume of sphere

8. Given $V = \pi r^2 h$, find $\frac{dV}{dt}$. $= 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$

volume of a cylinder

9. Given $V = \frac{1}{3}\pi r^2 h$, find $\frac{dV}{dt} = \frac{1}{3}\pi\left(2r \frac{dr}{dt} h + r^2 \frac{dh}{dt}\right)$

volume of a cone

10. Given $c^2 = a^2 + b^2$, find $\frac{dc}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{c}$

Pythagorean Thm

$c^2 = a^2 + b^2$

$\cancel{2}c \frac{dc}{dt} = \cancel{2}a \frac{da}{dt} + \cancel{2}b \frac{db}{dt}$

$\frac{dc}{dt} = \frac{a\frac{da}{dt} + b\, db/dt}{c}$

Now for our very first related rates problem….

11. Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

$\frac{dr}{dt} = 2 \text{ ft/s}$

$r = 60 \text{ ft}$

$\frac{dA}{dt} = ?$

$A = \pi r^2$

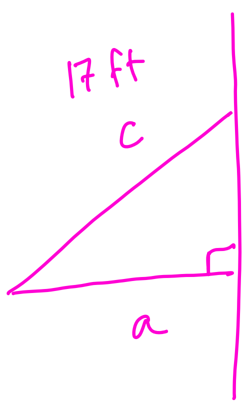
$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(60)(2)$

$\frac{dA}{dt} = 240\pi \text{ ft}^2/\text{s}$

12. A 17 ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5ft/s. How fast will the top of the ladder be moving down the wall when it is 8ft above the ground?

$\frac{15(5)}{8}$ ft/s



$b = 8 \text{ ft}$

$c = 17 \text{ ft}$

$\frac{da}{dt} = 5 \text{ ft/s}$

$\frac{db}{dt} =$

$b = 8 \text{ ft}$

$a^2 + b^2 = c^2$

$\cancel{2}a\frac{da}{dt} + \cancel{2}b\frac{db}{dt} = \cancel{2}c\frac{dc}{dt}$

$15(5) + 8\frac{db}{dt} = 17(0)$

$15(5) + 8\frac{db}{dt} = 0$

$8\frac{db}{dt} = -15(5)$

$\frac{db}{dt} = \frac{-15(5)}{8}$ ft/s

when $b = 8$

$a^2 + 8^2 = 17^2$

$a^2 + 64 = 289$

$a^2 = 225$

$a \cancel{^2} = 15$

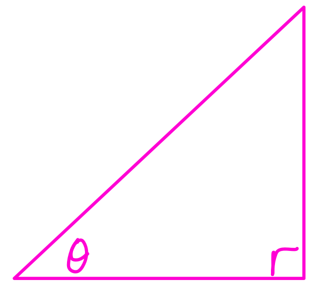
13. A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift out point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of .14 radians per minute. How fast is the balloon rising at that moment?

$a = 500\text{ft}$

$\frac{d\theta}{dt} = .14$ radians/m

$\theta = \frac{\pi}{4}$

$\frac{dh}{dt} =$



$h = 500\text{ft}$

a

$\tan\theta = \frac{h}{a}$

$a\tan\theta = h$

$a\sec^2\theta \frac{d\theta}{dt} + \frac{da}{dt}\tan\theta = \frac{dh}{dt}$

$500\sec^2(\frac{\pi}{4})(.14) + 0 \cancel{\tan\frac{\pi}{4}} = \frac{dh}{dt}$

$dh/dt = 500\left(\frac{2}{\sqrt{2}}\right)^2(.14)$ ft/min

cleaned up version

$500(2)(.14)$

$1000(.14)$

140 ft/min

AP Calculus AB: Related Rates Intro Homework

1. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing at the end of 10s?

Inverse Function Theorem

Homework 11-13

① $f(x) = x^3 + x$ $\qquad$ $f'(x) = 3x^2+1$

$2 = x^3 + x$ $\qquad$ $h'(2) = \dfrac{1}{3(1)^2+1} = \dfrac{1}{3+1} = \dfrac{1}{4}$ (B)

$x^3 + x - 2 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 1 & -2 \\ & & 1 & 1 & 2 \\ \hline & 1 & 1 & 2 & 0 \end{array}$$

* When finding roots using synthetic division on AP exam a good place to start is $\pm 1, \pm 2, 0$. Remember if $f(x) = 0$, then x is a root *

$x^2 + x + 2 = 0$

$x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$

$x = \dfrac{-1 \pm \sqrt{-7}}{2}$ imaginary

② $f(x) = x^2 + 1,\ x > 0$ $\quad$ $f[g(x)] = x$ $\quad$ means f and g are inverses of each other

a) find $g'(10)$.

Options

① find inverse of $f(x)$ $\qquad$ or $\qquad$ ② use new formula

$y = x^2 + 1$ $\qquad$ $f(x) = x^2 + 1$

$x = y^2 + 1$ $\qquad$ $10 = x^2 + 1$

$x - 1 = y^2$ $\qquad$ $9 = x^2$

$\sqrt{x-1} = y = g(x)$ $\qquad$ $3 = x$ (not -3 b/c of original restriction)

Now find $g'(x)$ $\qquad$ $f'(x) = 2x$

$g(x) = (x-1)^{1/2}$ $\qquad$ $f'(3) = 2(3) = 6$

$g'(x) = \frac{1}{2}(x-1)^{-1/2}$

$g'(x) = \dfrac{1}{2\sqrt{x-1}}$, $g'(10) = \dfrac{1}{2\sqrt{10-1}} = \boxed{\dfrac{1}{6}}$ $\qquad$ $g'(3) = \dfrac{1}{2(3)} = \dfrac{1}{6}$

2) (b)

$3 = x^2 + 1$ $\quad$ $f'(x) = 2x$

$2 = x^2$ $\quad$ $f'(\sqrt{2}) = 2\sqrt{2}$

$\sqrt{2} = x$

$$g'(3) = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{4}}$$

③ $f(x) = x^3 - 2x + 1$, $g(x) = f^{-1}(x)$

$x = 1$

(a) $x^3 - 2x + 1 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & -1 \\ \hline & 1 & 1 & -1 & 0 \end{array}$$

$f'(x) = 3x^2 - 2$

$f'(1) = 3(1)^2 - 2 = 1$

$$\boxed{g'(0) = \frac{1}{1} = 1}$$

$x^2 + x - 1 = 0$

$d = 1^2 - 4(1)(-1) = 1^2 + 4 = 5$

$x = \frac{-1 \pm \sqrt{5}}{2}$

* should have restrictions on original function
technically there are three separate answers here

(b) $x^3 - 2x + 1 = 5$

$x^3 - 2x - 4 = 0$ $\quad$ $x = 2$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -2 & -4 \\ & & 2 & 4 & 4 \\ \hline & 1 & 2 & 2 & 0 \end{array}$$

$f'(x) = 3x^2 - 2$

$f'(2) = 3(2)^2 - 2 = 10$

$x^2 + 2x + 2 = 0$

$$\boxed{g'(5) = \frac{1}{10}}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

④ $f(x) = \sin x + \cos x, \quad 0 \le x \le \pi \quad g = f^{-1}$, find $g'(-1) = \frac{1}{f'(x)} = \frac{1}{f'(\pi)}$

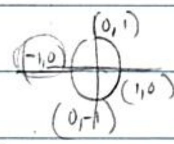
$^*\ \sin x + \cos x = -1$

$\sin x + \cos x + 1 = 0$

$f'(x) = \cos x - \sin x$

$f'(\pi) = \cos \pi - \sin \pi = -1 - 0 = -1$

$\quad\quad\quad\quad -1 \quad - \quad 0$



$\boxed{g'(-1) = \frac{1}{-1} = -1}$

$x = \pi$

$g'(-1) = \frac{1}{f'(\pi)}$

⑤ $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x + 6 \quad g = f^{-1} \quad g'(6)$

$x^3 - 6x^2 + 11x + 6 = 6$

$x^3 - 6x^2 + 11x = 0$

$x(x^2 - 6x + 11) = 0$

$x = 0 \quad\quad x = \frac{6 \pm \sqrt{36 - 4(1)(11)}}{2(1)}$ imaginary

$f'(x) = 3x^2 - 12x + 11$

$f'(0) = 3(0)^2 - 12(0) + 11 = 11$

$g'(6) = \frac{1}{11}$

⑥ $f(x) = 5x^2 + 1, \quad x \ge 0 \quad g = f^{-1}$, find $g'(11)$

$5x^2 + 1 = 11$

$5x^2 - 10 = 0$

$5(x^2 - 2) = 0$

$x = \pm\sqrt{2}$

$x = \sqrt{2}$ b/c of orig. rest.

$f'(x) = 10x$

$f'(\sqrt{2}) = 10\sqrt{2}$

$g'(11) = \frac{1}{10\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{20}$

7) $F(x) = 3x^2 - x,\ x > 1$ $\quad F(t(x)) = x$, $t'(10)$?

$3x^2 - x = 10$

$3x^2 - x - 10 = 0$

$(3x + 5)(x - 2)$

$x = -5/3$ $\quad x = 2$

reject b/c $x > 1$

$F'(x) = 6x - 1$

$F'(2) = 6(2) - 1 = 11$

$t'(10) = \frac{1}{11}$

restriction

x

(8) $f(x) = (x-2)\sqrt{x+1}$ g is inverse of f. Find $g'(18)$

$\left((x-2)\sqrt{x+1}\right)^2 = (18)^2$

$x^2 - 4x + 4\,(x+1) = 324$

$x^3 - 4x^2 + 4x + x^2 - 4x + 4 - 324 = 0$

$x^3 - 3x^2 - 320 = 0$

8	1	-3	0	-320
		8	40	+320
	1	5	40	0

$x^2 + 5x + 40 = 0$

$x = \frac{-5 \pm \sqrt{25 - 4(1)(40)}}{2(1)}$ imaginary

$x = 8$

$f'(x) = 1.(\sqrt{x+1}) + \frac{1}{2\sqrt{x+1}} (x-2)$

$f'(8) = 1(\sqrt{8+1}) + \frac{1}{2\sqrt{8+1}} \cdot 8-2$

$f'(8) = 3 + \frac{1}{6} \cdot 6$

$f'(8) = 3 + 1 = 4$

$g'(18) = \frac{1}{4}$

9) $f(x) = \frac{x}{x-2}$ $\quad (2, 4)$ is a point on $g(x)$ the inverse

so $(4, 2)$ is a point on $f(x)$

$2 = \frac{x}{x-2}$

$2x - 4 = x$

$-4 = -x$ ← *not needed*

$4 = x$

$g'(2) = \frac{1}{f'(4)}$

$f'(x) = \frac{(x-2)(1) - (x)(1)}{(x-2)^2} = \frac{x-2-x}{(x-2)^2} = \frac{-2}{(x-2)^2}$

$f'(4) = \frac{-2}{(4-2)^2} = \frac{-2}{4} = \frac{-1}{2}$

$g'(2) = \frac{1}{-1/2} = -2$

$y - 4 = -2(x - 2)$

$y - 4 = -2x + 4 \rightarrow y = -2x + 8$

⑩ $f(x) = 4$ when $x = 3$ $(?, 4)$ $(f^{-1})'(4) = \frac{1}{f'(x)}$

$f'(3) = 1$

So derivative of f^{-1} at 4 is $\frac{1}{1} = 1$

⑪ $g(x) = 6$ when $x = 4$ $(, 6)$

$g'(4) = 2$

So derivative of g^{-1} at $x = 6$ is $\frac{1}{2}$

⑫ $g(x) = 2$ when $x = 3$ $(, 2)$

$g'(3) = 0$

So the derivative of g^{-1} at $x = 2$ is not defined