Name: ______________________________ Date: ___________

AP Calculus AB L'Hôpital's Rule Ms. Loughran

Do Now:

Evaluate each of the following

$\frac{0}{0}$ 1. $\lim_{x\to 2}\frac{x^2-4}{x-2}$ $(\cancel{x-2})(x+2)$ $= 4$ L'Hopital's $\lim_{x\to 2}\frac{2x}{1} = 4$

$\frac{0}{0}$ 2. $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$ $(\cancel{\sqrt{x}-1})(\sqrt{x}+1)$ $= 2$ $\lim_{x\to 1}\frac{1}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to 1} 2x^{\frac{1}{2}} = 2$

3. $\lim_{x\to 2}\frac{x^2-4}{x+2}$ $= \frac{0}{4} = 0$ ← this is not a candidate for L'Hopital's rule

plug in

L'Hôpital's Rule

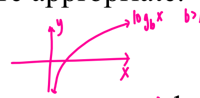
Let f and g be differentiable functions on some interval containing a (except possibly at a) such that $g'(x)\neq 0$ when $x\neq a$. Then, if $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$ (or $\pm\infty$), then

$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided that $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists or is $\pm\infty$.

Note:

- L'Hôpital's Rule also works if $x\to\pm\infty$
- Several consecutive applications of L'Hôpital's Rule, such as $\lim_{x\to a}\frac{f''(x)}{g''(x)}$, may be needed to evaluate a limit.
- You may NOT use L'Hôpital's Rule without $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.
- There is no guarantee that L'Hôpital's Rule will help. L'Hôpital's Rule helps only if we eventually obtain a limit which exists or is infinite.

Evaluate each limit. Use L'Hôpital's Rule where appropriate.



$\frac{0}{0}$ 1. $\lim\limits_{x\to2}\frac{x^3-8}{x^2-4} = \lim\limits_{x\to2}\frac{3x^2}{2x} = \frac{12}{4} = 3$

$\frac{0}{0}$ 2. $\lim\limits_{x\to0}\frac{\sin 5x}{x} = \lim\limits_{x\to0}\frac{5\cos 5x}{1} = 5(1) = 5$

$\frac{0}{0}$ 3. $\lim\limits_{x\to2}\frac{\sqrt{2+x}-2}{x-2} = \lim\limits_{x\to2}\frac{\frac{1}{2}(2+x)^{-\frac{1}{2}}}{1} = \frac{1}{2}(4)^{-\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

4. $\lim\limits_{x\to1}\frac{\sqrt[3]{x}-1}{x-1}$

5. $\lim\limits_{x\to2}\frac{x^2-4x+4}{x^3-12x+16}$

$\frac{0}{0}$ 6. $\lim\limits_{x\to\frac{\pi}{2}}\frac{1-\sin x}{1+\cos 2x} = \lim\limits_{x\to\frac{\pi}{2}}\frac{-\cos x}{-2\sin 2x} \; \left(\frac{0}{0}\right) = \lim\limits_{x\to\frac{\pi}{2}}\frac{\sin x}{-4\cos 2x} = \frac{1}{4}$

7. $\lim\limits_{x\to1}\frac{x^3-1}{4x^3-x-3}$

8. $\lim\limits_{x\to3}\frac{x-4}{x-2}$

9. $\lim\limits_{x\to0}\frac{x}{\tan x}$

10. $\lim\limits_{x\to1}\frac{1-\frac{1}{x}}{1-\frac{1}{x^2}}$

11. $\lim\limits_{x\to\infty}\frac{\log_2 x}{\log_3 x} \; \left(\frac{\infty}{\infty}\right) = \lim\limits_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{x\ln 3}} = \lim\limits_{x\to\infty}\frac{\cancel{x}\ln 3}{\cancel{x}\ln 2} = \frac{\ln 3}{\ln 2}$

12. $\frac{0}{0}$ $\lim\limits_{x\to0}\frac{e^{2x}-1}{\tan x} = \lim\limits_{x\to0}\frac{2e^{2x}}{\sec^2 x} = \lim\limits_{x\to0} 2e^{2x}\cos^2 x = 2(1)(1)^2 = 2$

13. $\lim\limits_{x\to0}\frac{\arctan x}{2x}$

14. $\lim\limits_{x\to\pi^+}\frac{2x-2\pi}{\sin(x-\pi)}$

15. $\lim\limits_{x\to0}\pi^2\frac{\tan 2x}{x\cos 2x}$

16. $\lim\limits_{x\to0}\frac{e^x-e^{-x}}{x}$

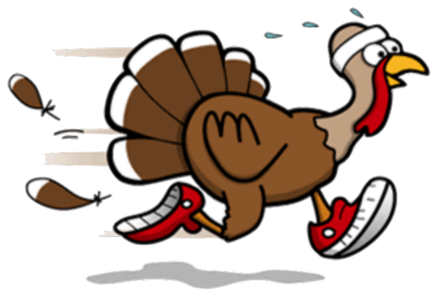
17. $\lim\limits_{x\to1}\frac{2\ln x}{x-1}$

18. $\lim\limits_{x\to0}\frac{3(e^x-e^{-x})}{\sin x}$

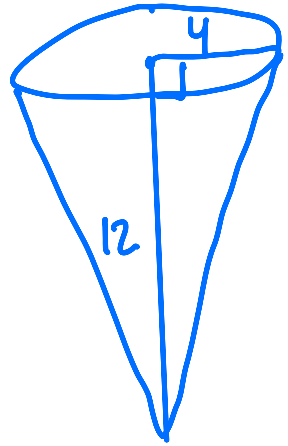
19. $\lim\limits_{x\to0}\frac{2x^2}{e^x-1-x}$

20. $\lim\limits_{x\to\infty}\frac{e^{2x}}{2x^2}$

Turkey Trot Solutions



Carman



$$\frac{r}{h} = \frac{4}{12}$$

$$\frac{r}{h} = \frac{1}{3}$$

$$3r = h$$

$$r = \frac{h}{3}$$

$$\frac{dV}{dt} = 1.5\,cm^3/s$$

$$\frac{dh}{dt} = ?$$

$$h = 5\,cm$$

James is filling an ice cream cone. The cone is 12 cm tall and has a radius of 4 cm. If the ice cream fills the cone evenly at a rate of 1.5 cm^3 / s, what is the rate of change of the height of the ice cream when the cone is filled up to 5 cm?

$$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 \cdot h$$

$$V = \frac{\pi}{27}h^3$$

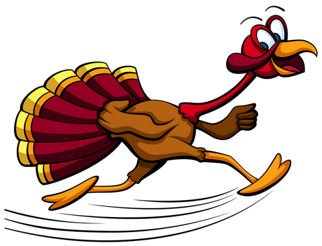
$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$$

$$\frac{3}{2} = \frac{\pi}{9}(5)^2\frac{dh}{dt}$$

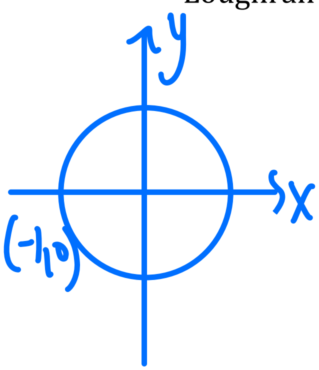
$$\frac{9}{25\pi}\cdot\frac{3}{2} = \frac{25\pi}{9}\frac{dh}{dt}\cdot\frac{9}{25\pi}$$

ANS: $-\frac{16x\cos^{-1}(4x^2)}{\sqrt{1-16x^4}}$

$$\frac{27}{50\pi}\,cm/s = \frac{dh}{dt}$$



Loughran



Find the slope of the tangent line to $\cos(\pi x) = x^7 y^2$ at the point $(-1,1)$.

$$-\sin(\pi x)\cdot\pi = 7x^6y^2 + x^7\cdot 2y\frac{dy}{dx}$$

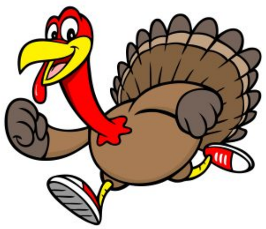
$$-\sin(-\pi)\cdot\pi = 7(-1)^6(1)^2 + (-1)^7\cdot 2(1)\frac{dy}{dx}$$

$$0\cdot\pi = 7 - 2\frac{dy}{dx}$$

ANS: $\frac{27}{50\pi}$

$$2\frac{dy}{dx} = 7$$

$$\frac{dy}{dx} = \frac{7}{2}$$



Barnett

$f'(x) = \cos x$

Given $f(g(x)) = x$ and $f(x) = \sin x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, find $g'\left(\frac{1}{2}\right)$.

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$

$g'\left(\frac{1}{2}\right) = \frac{1}{f'\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

ANS: $\frac{7}{2}$

Windwer

Given $y = \sin^{-1}\left(3x^5+1\right)^3$, find y'.

$$y' = \frac{1}{\sqrt{1-\left(\left(3x^5+1\right)^3\right)^2}} \cdot 3\left(3x^5+1\right)^2 \cdot 15x^4$$

ANS: $\frac{2}{\sqrt{3}}$

$$= \frac{45x^4\left(3x^5+1\right)^2}{\sqrt{1-\left(3x^5+1\right)^6}}$$

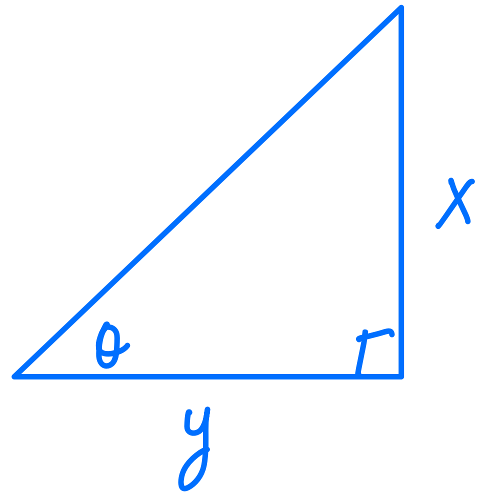
Lee

$\theta = \frac{\pi}{4}$

$y = 100m$

$\frac{dx}{dt} = -2m/s$

$\frac{d\theta}{dt} = ?$



$\tan\theta = \frac{x}{y}$

$y\tan\theta = x$

You are looking at the ball drop in Times Square on New Year's Eve at a distance of 100 m away from the base of the structure. If the ball drops at a constant rate of 2 m/s. what is the rate of change of the angle between you and the ball when the angle is $\frac{\pi}{4}$?

ANS: $\frac{45x^4(3x^5+1)^2}{\sqrt{1-(3x^5+1)^6}}$

$$y\sec^2\theta\frac{d\theta}{dt} + \frac{dy}{dt}\tan\theta = \frac{dx}{dt}$$

$$100\left(\sec^2\left(\frac{\pi}{4}\right)\right)\frac{d\theta}{dt} + \cancel{0\tan\left(\frac{\pi}{4}\right)} = -2$$

$$100\left(\frac{2}{\sqrt{2}}\right)^2\frac{d\theta}{dt} = -2$$

$$200\frac{d\theta}{dt} = -2$$

$$\frac{d\theta}{dt} = \frac{-1}{100} \text{ rad/s}$$

Jacknis

Find the derivative of $h(x) = \sin(\arccos(x))$.

$$h'(x) = \cos(\arccos x) \cdot -\frac{1}{\sqrt{1-x^2}}$$

$$x \cdot -\frac{1}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

ANS: $-\frac{1}{100}$

Valentino

$f(x) = \sqrt{x}$

$x = 3$

Find the exact value of $\lim_{h \to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h}$.

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(3) = \frac{1}{2\sqrt{3}}$

ANS: $-\frac{x}{\sqrt{1-x^2}}$

$$f'(x)=\begin{cases} 3ax^2 & x<\frac{1}{2} \\ 2bx & x\geq\frac{1}{2}\end{cases}$$

diff: $3a\left(\frac{1}{2}\right)^2 = 2b\left(\frac{1}{2}\right)$

$\frac{3}{4}a = b$

Stack

Cont: $a\left(\frac{1}{2}\right)^3 + 2 = b\left(\frac{1}{2}\right)^2 + 1$

$\frac{a}{8} + 2 = \frac{b}{4} + 1$

$a + 16 = 2b + 8$

$a = 2b - 8$

Given the function $f(x) = \begin{cases} ax^3 + 2, & x < \dfrac{1}{2} \\ bx^2 + 1, & x \geq \dfrac{1}{2} \end{cases}$

If the function is differentiable at $x = \dfrac{1}{2}$, then $a + b =$ 28

$a = 2\left(\frac{3}{4}a\right) - 8$

$a = \frac{3}{2}a - 8$

$2a = 3a - 16$

$-a = -16$

$a = 16$

$b = \frac{3}{4}a$

$b = \frac{3}{4}(16) = 12$

ANS: $\dfrac{1}{2\sqrt{3}}$

Ciavarella

0

Find the sum of the constants a, b, and c for which $y = ax^2 + bx + c$ passes through $(0,1)$ and is tangent to $y = x - 1$ at the point $(1,0)$.

$m = 1$

$(0,1)$ $\quad 1 = a(0)^2 + b(0) + c$

$1 = c$

$(1,0)$ $\quad 0 = a(1)^2 + b(1) + c$

$0 = a + b + 1$

$-1 = a + b$

$1 = -a - b$

$y' = 2ax + b$

$2ax + b = 1 \quad @\ (1,0)$

$2a(1) + b = 1$

$2a + b = 1$

ANS: 28

$2a + \cancel{b} = 1$

$-a - \cancel{b} = 1$

$a = 2$

$-1 = a + b$

$-1 = 2 + b$

$-3 = b$

Edelman

$$\text{If } \lim_{x\to\infty}\left(\frac{5n^3}{20-3n-kn^3}\right)=\frac{1}{2} \text{ then } k=$$

$$\frac{5}{-k}=\frac{1}{2}$$

$$-k=10$$

$$k=-10$$

ANS: 0

Callahan

$$\text{If } f(x) = \log_5\left(xe^x\right), \text{ find } f'(x).$$

$$f'(x) = \frac{1}{xe^x \ln 5} \cdot xe^x + e^x$$

ANS: -10

$$f'(x) = \frac{xe^x + e^x}{xe^x \ln 5} = \frac{\cancel{e^x}(x+1)}{x\cancel{e^x} \ln 5} = \frac{x+1}{x \ln 5}$$

Sherwood

$$\text{If } y = \frac{3^{2x}}{x}, \text{ find } y'.$$

$$y' = \frac{x \cdot 3^{2x} \ln 3 \cdot 2 - 3^{2x}}{x^2}$$

ANS: $\frac{x+1}{x \ln 5}$

$$y' = \frac{3^{2x}(2x \ln 3 - 1)}{x^2}$$

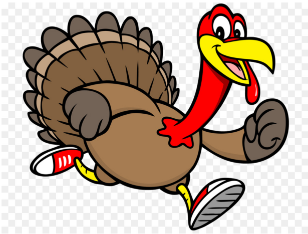
Sellers

$$\text{If } g(x) = \log_6(3x\tan x)\text{, find } g'(x).$$

$$g'(x) = \frac{1}{(3x\tan x)\ln 6} \cdot 3\tan x + 3x\sec^2 x$$

$$g'(x) = \frac{\cancel{3}(\tan x + x\sec^2 x)}{\cancel{3}x\tan x\ln 6}$$

ANS: $\dfrac{3^{2x}(2x\ln 3 - 1)}{x^2}$


Simon

$$\text{Given } y=\left(\cos^{-1}\left(4x^2\right)\right)^2\text{, find } \frac{dy}{dx}.$$

$$\frac{dy}{dx} = 2\cos^{-1}(4x^2)\cdot -\frac{1}{\sqrt{1-(4x^2)^2}}\cdot 8x$$

$$\frac{dy}{dx} = -\frac{16x\cos^{-1}(4x^2)}{\sqrt{1-16x^4}}$$

ANS: $\dfrac{x\sec^2 x+\tan x}{x\tan x\ln 6}$