Name: ______________________________ Date: ___________

AP Calc AB: Second Derivative Test Ms. Loughran

Do Now:

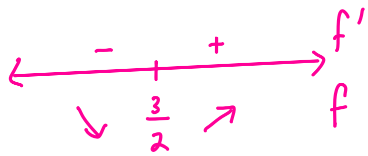
For 1and 2, locate the relative extrema of each. (where is it?)

1. $f(x) = x^2 - 3x + 3$

$f'(x) = 2x - 3$

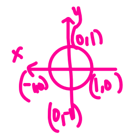
$2x - 3 = 0$

$x = \frac{3}{2}$



There is a relative minimum at $x = 3/2$

2. $f(x) = \cos x$

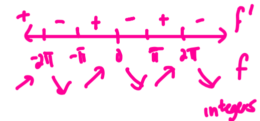


$f'(x) = -\sin x$

$-\sin x = 0$

$\sin x = 0$

$x = 0, \pi, 2\pi, 3\pi, -\pi, -2\pi, \ldots$



rel min: $x = \pi k$, $k \in$ odd $\mathbb{Z}$

rel max: $x = \pi n$, $n \in$ even $\mathbb{Z}$

3. Find the open interval on which f is concave up and on which f is concave down:

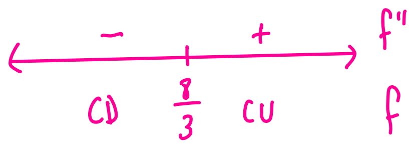
$f(x) = x^3 - 8x^2 + 5$

$f'(x) = 3x^2 - 16x$

$f''(x) = 6x - 16$

$6x - 16 = 0$

$x = \frac{16}{6} = \frac{8}{3}$



CD: $(-\infty, \frac{8}{3})$

CU: $(8/3, \infty)$

Second Derivative Test:

Suppose c is a critical point of a continuous function f

$f \cap$ **1.** If $f'(c) = 0$ and $f''(c) < 0$, then f has a local (relative) maximum at $x = c$.

$f \cup$ **2.** If $f'(c) = 0$ and $f''(c) > 0$, then f has a local (relative) minimum at $x = c$.

Note: If $f''(c) = 0$ you can not use the SDT also if $f''(c)$ dne you can not use the SDT. In these cases you would need to use the FDT.

Using the second derivative test, find the local extreme values of each.

1. $f(x)=x^3-12x-5$

still need critical values

$f'(x)=3x^2-12$

$3x^2-12=0$

$3x^2=12$

$x^2=4$

$x=\pm 2$

$f''(x)=6x$

$f(2)=2^3-12(2)-5$

$=8-24-5$

$f(-2)=-8+24-5$

SDT

$f''(2)>0$ therefore there is a rel. min of -21 at $x=2$

$f''(-2)<0$ therefore there is a rel. max of 11 at $x=-2$

2. $f(x)=3x-x^3+5$

$f'(x)=3-3x^2$

$3-3x^2=0$

$3=3x^2$

$1=x^2$

$\pm 1=x$

$f''(x)=-6x$

$f(1)=3-1+5$

$f(-1)=-3+1+5$

$f''(1)<0$ therefore there is a local max of 7 at $x=1$

$f''(-1)>0$ therefore there is a local min of 3 at $x=-1$

3. $f(x)=x^3+3x^2-2$

$f'(x)=3x^2+6x$

$3x^2+6x=0$

$3x(x+2)=0$

$x=0$ | $x=-2$

$f''(x)=6x+6$

$f''(0)>0$ therefore there is a rel min of -2 at $x=0$

$f''(-2)<0$ therefore there is a rel max of 2 at $x=-2$

4. $y = xe^x$

$y' = e^x + xe^x$

$0 = e^x(1+x)$

$e^x \neq 0$ | $x = -1$

$y'' = e^x(1) + (1+x)e^x$

$y'' = e^x(1+1+x)$

$y'' = e^x(x+2)$

SDT

$y''(-1) = e^{-1}(-1+2) > 0$

therefore there is a rel min of $-e^{-1}$ or $-\frac{1}{e}$ at $x = -1$

5. $y = x^5 - 80x + 100$

$y' = 5x^4 - 80$

$5x^4 - 80 = 0$

$x^4 - 16 = 0$

$(x^2-4)(x^2+4) = 0$

$x = \pm 2$ | ~~$x = \pm 2i$~~

$y'' = 20x^3$

$\cup$ $y''(2) > 0$

$\cap$ $y''(-2) < 0$

There is a rel min of -28 at 2

There is a rel max of 228 at -2

6. $y = 3x^5 - 25x^3 + 60x + 20$

$y' = 15x^4 - 75x^2 + 60$

$15x^4 - 75x^2 + 60 = 0$

$15(x^4 - 5x^2 + 4) = 0$

$15(x^2-4)(x^2-1) = 0$

$x = \pm 2, \pm 1$

$y'' = 60x^3 - 150x$

$y''(2) > 0$ there is a rel min of 36 at $x=2$

$y''(-2) < 0$ there is a rel max of 4 at $x=-2$

$y''(1) < 0$ there is a rel max of 58 at $x=1$

$y''(-1) > 0$ there is a rel min -18 at $x=-1$

7. $y = xe^{-x}$

$y' = -xe^{-x} + e^{-x}$

$-xe^{-x} + e^{-x} = 0$

$-e^{-x}(x-1) = 0$

$-e^{-x} \neq 0$	$x = 1$

$(x-1)(e^{-x})$

$y'' = -e^{-x} + (x-1)(-e^{-x})(-1)$

$y'' = e^{-x}(-1 + x - 1)$

$y'' = e^{-x}(x-2)$

$y''(1) < 0$

there is a rel max of e^{-1} or $\frac{1}{e}$

at $x=1$

WHEN f, f', and f" CHANGE SIGN: TESTS FOR LOCAL EXTREMA & INFLECTION POINTS

When graphing a function f, find where $f(x) = 0$, where $f'(x) = 0$, and where $f''(x) = 0$, if the solutions are easy. Determine where $f'(x)$ is positive and where it is negative. Determine also where $f''(x)$ is positive and where it is negative. The following table contrasts the interpretations of the signs of f, f', and f''. (It is assumed that f, $f^{(1)}$, and $f^{(2)}$ are continuous.)

	Is Positive	*Is Negative*	*Changes Sign*
Where the ordinate $f(x)$	The graph is above the x axis	The graph is below the x axis	The graph crosses the x axis
Where the slope $f'(x)$	The graph slopes upward	The graph slopes downward	The graph has a horizontal tangent and a relative maximum or minimum
Where $f''(x)$	The graph is concave upward (like a cup)	The graph is concave downward	The graph has an inflection point

Keep in mind that the graph can have an inflection point at x_0, even though the second derivative is not defined at x_0 (Example 3). Similarly, a graph can have a maximum or minimum at x_0, even though the first derivative is not defined at x_0. (Consider $f(x) = |x|$ at $x_0 = 0$.)

The second derivative is also useful in searching for relative maxima or minima. For instance, let a be a critical number for the function f and assume that $f''(a)$ happens to be negative. If f'' is continuous in some open interval that contains a, then $f''(a)$ remains negative for a suitably small open interval that contains a. This means that the graph of f is concave downward near $(a, f(a))$, hence lies below its tangent lines. In particular, it lies below the horizontal tangent line at the critical point $(a, f(a))$. Thus the function has a *relative maximum* at the critical number a. This observation suggests the following test for a relative maximum or minimum.

Second-derivative test for local maximum or minimum. Let f be a function with continuous derivative and second derivative. Let a be a critical number for f, that is, $f'(a) = 0$.

If $f''(a) < 0$, f has a local maximum at a.
If $f''(a) > 0$, f has a local minimum at a.

What f ' and f" Tell us About f

Assume f is defined everywhere.

Behavior of f ' ⇒ (implies)	**Behavior of f**
$f' > 0$ on an interval	f is increasing on the interval
$f' < 0$ on an interval	f is decreasing on the interval
$f'(c) = 0$	f has a horizontal tangent at $x = c$
$f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$ **That is, f' changes from negative to positive at c**	f has a relative minimum at $x = c$
$f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ **That is, f' changes from positive to negative at c**	f has a relative maximum at $x = c$
f ' increasing (or $f''>0$) on an interval	f is concave up on the interval
f ' decreasing (or $f''<0$) on an interval	f is concave down on the interval
f" changes sign at c ALSO f' goes from increasing to decreasing, or vice versa, at c	f has an inflection point at x=c

- In proving a relative minimum or maximum, it is never enough to show that the derivative is zero. A single example demonstrates this: Consider $C(x) = x^3$ for $x = 0$. The derivative $C'(x) = 3x^2$ so $C'(0) = 0$ but the function has neither a minimum or maximum there.
- To prove a relative maximum or minimum when $x = c$, it is always necessary to do one of these things:

  Show that $f'(x)$ changes sign at x=c (i.e use the First Derivative Test)

  Or

  Show that $f'(c)=0$ and $f''(c) \neq 0$ (i.e. use the Second Derivative Test)
- Values of x where $f'(x) = 0$ or $f'(x)$ is undefined are called critical numbers. These are merely candidates for x-values of a maximum or minimum – you must still see if $f'(x)$ changes sign.
- Even well-labeled sign charts are not enough to show extrema and inflection points. You must state the link between f' and f. See "Lessons Learned at the 2005 Readings."
- If the problem asks for an absolute extreme, you must also evaluate the function at the endpoints of the interval, and compare.

Homework 12-11

CALCULUS AB
SECTION I, Part A
Time—55 minutes
Number of questions—28

A CALCULATOR MAY NOT BE USED ON THIS PART OF THE EXAM.

Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding oval on the answer sheet. No credit will be given for anything written in the exam book. Do not spend too much time on any one problem.

In this exam:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1}x = \arcsin x$).

1. $\lim_{x \to \infty} \frac{(2x-1)(3-x)}{(x-1)(x+3)}$ is

(A) -3 (B) -2 (C) 2 (D) 3 (E) nonexistent

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2. $\int \frac{1}{x^2}\, dx =$

(A) $\ln x^2 + C$ (B) $-\ln x^2 + C$ (C) $x^{-1} + C$ (D) $-x^{-1} + C$ (E) $-2x^{-3} + C$

3. If $f(x)=(x-1)\left(x^2+2\right)^3$, then $f'(x)=$

(A) $6x\left(x^2+2\right)^2$

(B) $6x(x-1)\left(x^2+2\right)^2$

(C) $\left(x^2+2\right)^2\left(x^2+3x-1\right)$

(D) $\left(x^2+2\right)^2\left(7x^2-6x+2\right)$

(E) $-3(x-1)\left(x^2+2\right)^2$

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4. $\int(\sin(2x)+\cos(2x))\,dx=$

(A) $\frac{1}{2}\cos(2x)+\frac{1}{2}\sin(2x)+C$

(B) $-\frac{1}{2}\cos(2x)+\frac{1}{2}\sin(2x)+C$

(C) $2\cos(2x)+2\sin(2x)+C$

(D) $2\cos(2x)-2\sin(2x)+C$

(E) $-2\cos(2x)+2\sin(2x)+C$

5. $\lim\limits_{x \to 0} \dfrac{5x^4 + 8x^2}{3x^4 - 16x^2}$ is

(A) $-\dfrac{1}{2}$ (B) 0 (C) 1 (D) $\dfrac{5}{3}$ (E) nonexistent

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$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

6. Let f be the function defined above. Which of the following statements about f are true?

I. f has a limit at $x = 2$.

II. f is continuous at $x = 2$.

III. f is differentiable at $x = 2$.

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I, II, and III

7. A particle moves along the x-axis with velocity given by $v(t) = 3t^2 + 6t$ for time $t \geq 0$. If the particle is at position $x = 2$ at time $t = 0$, what is the position of the particle at time $t = 1$?

(A) 4 (B) 6 (C) 9 (D) 11 (E) 12

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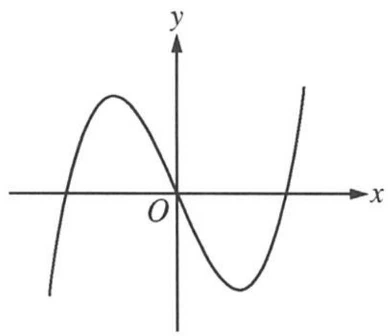
8. If $f(x) = \cos(3x)$, then $f'\left(\frac{\pi}{9}\right) =$

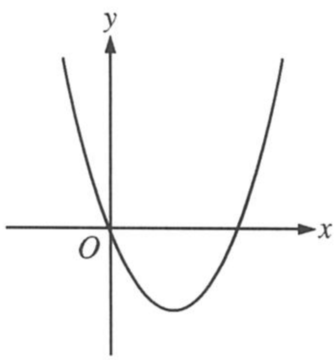
(A) $\frac{3\sqrt{3}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $-\frac{\sqrt{3}}{2}$ (D) $-\frac{3}{2}$ (E) $-\frac{3\sqrt{3}}{2}$



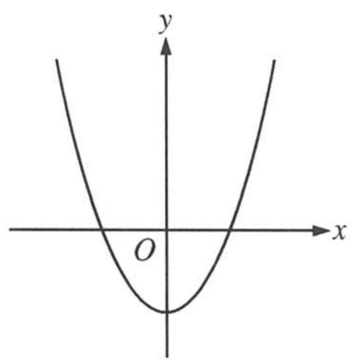
Graph of f

11. The graph of a function f is shown above. Which of the following could be the graph of f', the derivative of f?

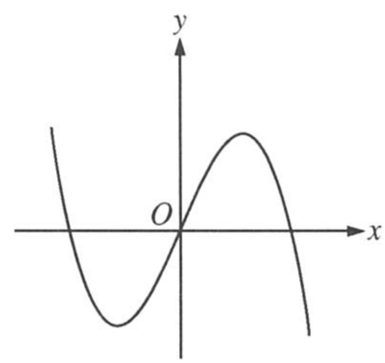
(A)



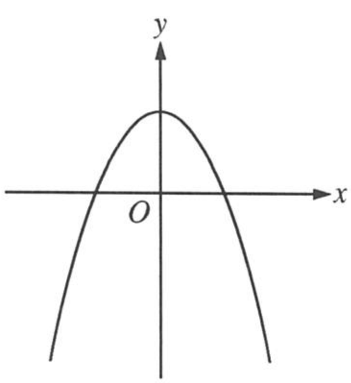
(B)



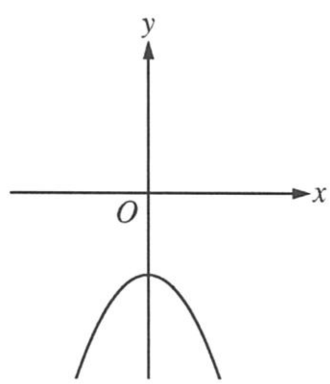
(C)



(D)



(E)



12. If $f(x) = e^{(2/x)}$, then $f'(x) =$

(A) $2e^{(2/x)} \ln x$ (B) $e^{(2/x)}$ (C) $e^{\left(-2/x^2\right)}$ (D) $-\frac{2}{x^2}e^{(2/x)}$ (E) $-2x^2e^{(2/x)}$

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13. If $f(x) = x^2 + 2x$, then $\frac{d}{dx}\left(f(\ln x)\right) =$

(A) $\frac{2\ln x + 2}{x}$ (B) $2x\ln x + 2x$ (C) $2\ln x + 2$ (D) $2\ln x + \frac{2}{x}$ (E) $\frac{2x + 2}{x}$

x	0	1	2	3
$f''(x)$	5	0	–7	4

14. The polynomial function f has selected values of its second derivative f'' given in the table above. Which of the following statements must be true?

(A) f is increasing on the interval $(0, 2)$.

(B) f is decreasing on the interval $(0, 2)$.

(C) f has a local maximum at $x = 1$.

(D) The graph of f has a point of inflection at $x = 1$.

(E) The graph of f changes concavity in the interval $(0, 2)$.

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15. $\int \frac{x}{x^2 - 4}\, dx =$

(A) $\frac{-1}{4\left(x^2 - 4\right)^2} + C$

(B) $\frac{1}{2\left(x^2 - 4\right)} + C$

(C) $\frac{1}{2}\ln\left|x^2 - 4\right| + C$

(D) $2\ln\left|x^2 - 4\right| + C$

(E) $\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

16. If $\sin(xy) = x$, then $\dfrac{dy}{dx} =$

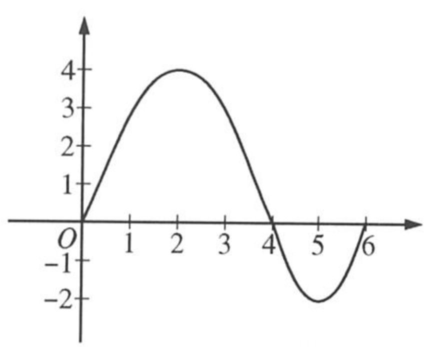
(A) $\dfrac{1}{\cos(xy)}$

(B) $\dfrac{1}{x\cos(xy)}$

(C) $\dfrac{1-\cos(xy)}{\cos(xy)}$

(D) $\dfrac{1-y\cos(xy)}{x\cos(xy)}$

(E) $\dfrac{y(1-\cos(xy))}{x}$



Graph of f

17. The graph of the function f shown above has horizontal tangents at $x = 2$ and $x = 5$. Let g be the function defined by $g(x) = \int_0^x f(t)\,dt$. For what values of x does the graph of g have a point of inflection?

(A) 2 only (B) 4 only (C) 2 and 5 only (D) 2, 4, and 5 (E) 0, 4, and 6

18. In the xy-plane, the line $x + y = k$, where k is a constant, is tangent to the graph of $y = x^2 + 3x + 1$. What is the value of k ?

(A) -3 (B) -2 (C) -1 (D) 0 (E) 1

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19. What are all horizontal asymptotes of the graph of $y = \frac{5 + 2^x}{1 - 2^x}$ in the xy-plane?

(A) $y = -1$ only

(B) $y = 0$ only

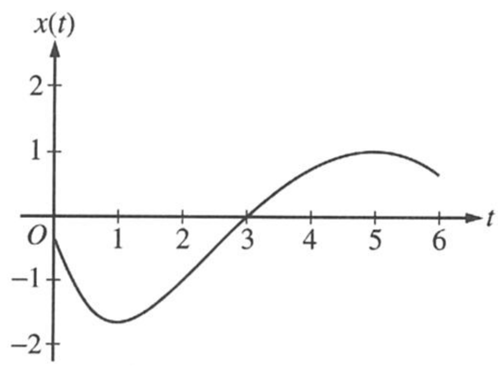
(C) $y = 5$ only

(D) $y = -1$ and $y = 0$

(E) $y = -1$ and $y = 5$

20. Let f be a function with a second derivative given by $f''(x) = x^2(x-3)(x-6)$. What are the x-coordinates of the points of inflection of the graph of f?

(A) 0 only (B) 3 only (C) 0 and 6 only (D) 3 and 6 only (E) 0, 3, and 6



21. A particle moves along a straight line. The graph of the particle's position $x(t)$ at time t is shown above for $0 < t < 6$. The graph has horizontal tangents at $t = 1$ and $t = 5$ and a point of inflection at $t = 2$. For what values of t is the velocity of the particle increasing?

(A) $0 < t < 2$

(B) $1 < t < 5$

(C) $2 < t < 6$

(D) $3 < t < 5$ only

(E) $1 < t < 2$ and $5 < t < 6$

$$f(x) = \begin{cases} cx + d & \text{for } x \le 2 \\ x^2 - cx & \text{for } x > 2 \end{cases}$$

25. Let f be the function defined above, where c and d are constants. If f is differentiable at $x = 2$, what is the value of $c + d$?

(A) -4 (B) -2 (C) 0 (D) 2 (E) 4

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26. What is the slope of the line tangent to the curve $y = \arctan(4x)$ at the point at which $x = \frac{1}{4}$?

(A) 2 (B) $\frac{1}{2}$ (C) 0 (D) $-\frac{1}{2}$ (E) -2

28. Let f be a differentiable function such that $f(3) = 15$, $f(6) = 3$, $f'(3) = -8$, and $f'(6) = -2$. The function g is differentiable and $g(x) = f^{-1}(x)$ for all x. What is the value of $g'(3)$?

(A) $-\frac{1}{2}$

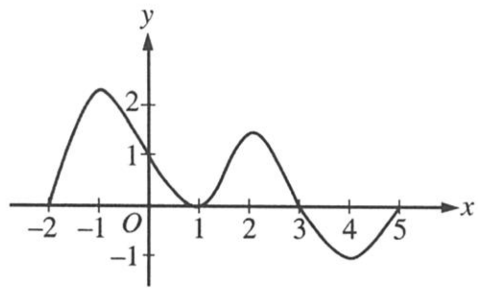
(B) $-\frac{1}{8}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

(E) The value of $g'(3)$ cannot be determined from the information given.

END OF PART A OF SECTION I



Graph of f'

76. The graph of f', the derivative of f, is shown above for $-2 \le x \le 5$. On what intervals is f increasing?

(A) $[-2, 1]$ only

(B) $[-2, 3]$

(C) $[3, 5]$ only

(D) $[0, 1.5]$ and $[3, 5]$

(E) $[-2, -1]$, $[1, 2]$, and $[4, 5]$