Name:_____ Date:_____

AP Calculus: The Definite Integral as an Average Ms. Loughran

We know how to find the average of n numbers: add them and divide by n. But how do we find the average value of a continuously varying function? Let us consider an example. Suppose $C = f(t)$ is the temperature at time t, measured in hours since midnight, and that we want to calculate the average temperature over a 24-hour period. One way to start would be to average the temperatures at n times, $t_1, t_2, \ldots, t_n$, during the day.

$$\text{Average temperature} \approx \frac{f(t_1) + f(t_2) + \cdots + f(t_n)}{n}.$$

The larger we make n, the better the approximation. We can rewrite this expression as a Riemann sum over the interval $0 \le t \le 24$ if we use the fact that $\Delta t = 24/n$, so $n = 24/\Delta t$:

$$\begin{aligned}\text{Average temperature} &\approx \frac{f(t_1) + f(t_2) + \cdots + f(t_n)}{24/\Delta t} \\ &= \frac{f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t}{24} \\ &= \frac{1}{24}\sum_{i=1}^{n} f(t_i)\Delta t.\end{aligned}$$

As $n \to \infty$, the Riemann sum tends towards an integral and also approximates the average temperature better. Thus, in the limit

$$\begin{aligned}\text{Average temperature} &= \lim_{n\to\infty} \frac{1}{24}\sum_{i=1}^{n} f(t_i)\Delta t \\ &= \frac{1}{24}\int_0^{24} f(t)\,dt.\end{aligned}$$

Thus we have found a way of expressing the average temperature in terms of an integral. Generalizing for any function f, we define

$$\text{Average value of } f \text{ from } a \text{ to } b = \frac{1}{b-a}\int_a^b f(x)\,dx$$

How to Visualize the Average on a Graph

The definition of average value tells us that

$$(\text{Average value of } f)\cdot(b-a) = \int_a^b f(x)\,dx.$$

Thus, if we interpret the integral as the area under the graph of f, then we can think of the average value of f as the height of the rectangle with the same area that is on the same base, $(b-a)$. (See Figure 3.18.)

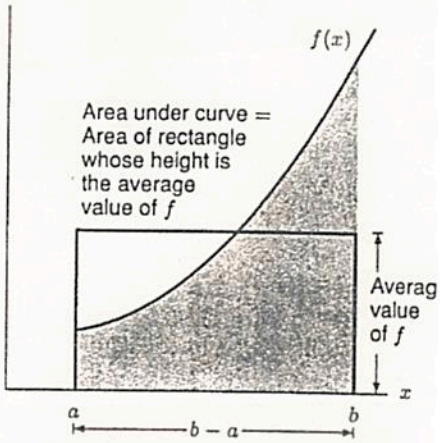

Figure 3.18: Area and average value

Practice

1. Find the average value of $f(x)=x^2$ from $x=2$ to $x=4$.

2. Find the average value of $f(x)=\sqrt{x}$ from $x=0$ to $x=16$.

3. Find the average value of $f(x)=e^{2x}$ on the interval [-1,1].

4. Find the average value of $f(x)=\cos x$ from $x=0$ to $x=\pi$.

5. Find the average velocity of $v(t)=t^2-2$ on the interval [-2,3].