31. $\dfrac{x^5 + 3x^3 - 6}{x - 1}$

32. $\dfrac{x^3 - 9x^2 + 27x - 27}{x - 3}$

33. $\dfrac{2x^3 + 3x^2 - 2x + 1}{x - \frac{1}{2}}$

34. $\dfrac{6x^4 + 10x^3 + 5x^2 + x + 1}{x + \frac{2}{3}}$

35. $\dfrac{x^3 - 27}{x - 3}$

36. $\dfrac{x^4 - 16}{x + 2}$

37–49 ■ Use synthetic division and the Remainder Theorem to evaluate $P(c)$.

37. $P(x) = 4x^2 + 12x + 5, \quad c = -1$

38. $P(x) = 2x^2 + 9x + 1, \quad c = \frac{1}{2}$

39. $P(x) = x^3 + 3x^2 - 7x + 6, \quad c = 2$

40. $P(x) = x^3 - x^2 + x + 5, \quad c = -1$

41. $P(x) = x^3 + 2x^2 - 7, \quad c = -2$

42. $P(x) = 2x^3 - 21x^2 + 9x - 200, \quad c = 11$

43. $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14, \quad c = -7$

44. $P(x) = 6x^5 + 10x^3 + x + 1, \quad c = -2$

45. $P(x) = x^7 - 3x^2 - 1, \quad c = 3$

46. $P(x) = -2x^6 + 7x^5 + 40x^4 - 7x^2 + 10x + 112, \quad c = -3$

47. $P(x) = 3x^3 + 4x^2 - 2x + 1, \quad c = \frac{2}{3}$

48. $P(x) = x^3 - x + 1, \quad c = \frac{1}{4}$

49. $P(x) = x^3 + 2x^2 - 3x - 8, \quad c = 0.1$

50. Let

$$P(x) = 6x^7 - 40x^6 + 16x^5 - 200x^4 - 60x^3 - 69x^2 + 13x - 139$$

Calculate $P(7)$ by **(a)** using synthetic division and **(b)** substituting $x = 7$ into the polynomial and evaluating directly.

51–54 ■ Use the Factor Theorem to show that $x - c$ is a factor of $P(x)$ for the given value(s) of c.

51. $P(x) = x^3 - 3x^2 + 3x - 1, \quad c = 1$

52. $P(x) = x^3 + 2x^2 - 3x - 10, \quad c = 2$

53. $P(x) = 2x^3 + 7x^2 + 6x - 5, \quad c = \frac{1}{2}$

54. $P(x) = x^4 + 3x^3 - 16x^2 - 27x + 63, \quad c = 3, -3$

55–56 ■ Show that the given value(s) of c are zeros of $P(x)$, and find all other zeros of $P(x)$.

55. $P(x) = x^3 - x^2 - 11x + 15, \quad c = 3$

56. $P(x) = 3x^4 - x^3 - 21x^2 - 11x + 6, \quad c = \frac{1}{3}, -2$

57–60 ■ Find a polynomial of the specified degree that has the given zeros.

57. Degree 3; zeros $-1, 1, 3$

58. Degree 4; zeros $-2, 0, 2, 4$

59. Degree 4; zeros $-1, 1, 3, 5$

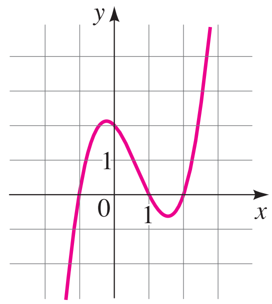
60. Degree 5; zeros $-2, -1, 0, 1, 2$

61. Find a polynomial of degree 3 that has zeros $1, -2$, and 3, and in which the coefficient of x^2 is 3.

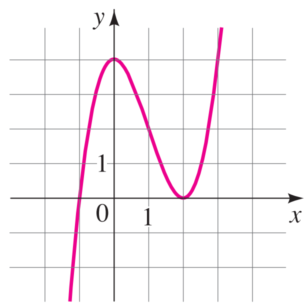
62. Find a polynomial of degree 4 that has integer coefficients and zeros $1, -1, 2$, and $\frac{1}{2}$.

63–66 ■ Find the polynomial of the specified degree whose graph is shown.

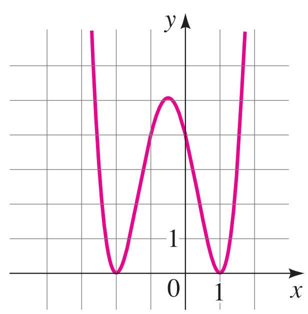
63. Degree 3



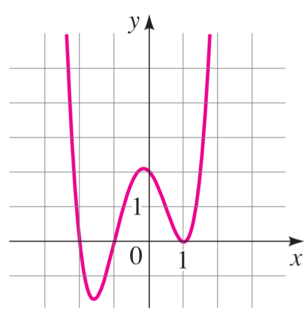
64. Degree 3



65. Degree 4



66. Degree 4



Discovery • Discussion

67. Impossible Division? Suppose you were asked to solve the following two problems on a test:

A. Find the remainder when $6x^{1000} - 17x^{562} + 12x + 26$ is divided by $x + 1$.

B. Is $x - 1$ a factor of $x^{567} - 3x^{400} + x^9 + 2$?

Obviously, it's impossible to solve these problems by dividing, because the polynomials are of such large degree. Use one or more of the theorems in this section to solve these problems *without* actually dividing.

Example 8 Factoring a Polynomial into Linear and Quadratic Factors

Let $P(x) = x^4 + 2x^2 - 8$.

(a) Factor P into linear and irreducible quadratic factors with real coefficients.

(b) Factor P completely into linear factors with complex coefficients.

Solution

(a)

$$\begin{aligned} P(x) &= x^4 + 2x^2 - 8 \\ &= (x^2 - 2)(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \end{aligned}$$

The factor $x^2 + 4$ is irreducible since it has only the imaginary zeros $\pm 2i$.

(b) To get the complete factorization, we factor the remaining quadratic factor.

$$\begin{aligned} P(x) &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x - 2i)(x + 2i) \end{aligned}$$

■

3.5 Exercises

1–12 ■ A polynomial P is given.

(a) Find all zeros of P, real and complex.

(b) Factor P completely.

1. $P(x) = x^4 + 4x^2$

2. $P(x) = x^5 + 9x^3$

3. $P(x) = x^3 - 2x^2 + 2x$

4. $P(x) = x^3 + x^2 + x$

5. $P(x) = x^4 + 2x^2 + 1$

6. $P(x) = x^4 - x^2 - 2$

7. $P(x) = x^4 - 16$

8. $P(x) = x^4 + 6x^2 + 9$

9. $P(x) = x^3 + 8$

10. $P(x) = x^3 - 8$

11. $P(x) = x^6 - 1$

12. $P(x) = x^6 - 7x^3 - 8$

13–30 ■ Factor the polynomial completely and find all its zeros. State the multiplicity of each zero.

13. $P(x) = x^2 + 25$

14. $P(x) = 4x^2 + 9$

15. $Q(x) = x^2 + 2x + 2$

16. $Q(x) = x^2 - 8x + 17$

17. $P(x) = x^3 + 4x$

18. $P(x) = x^3 - x^2 + x$

19. $Q(x) = x^4 - 1$

20. $Q(x) = x^4 - 625$

21. $P(x) = 16x^4 - 81$

22. $P(x) = x^3 - 64$

23. $P(x) = x^3 + x^2 + 9x + 9$

24. $P(x) = x^6 - 729$

25. $Q(x) = x^4 + 2x^2 + 1$

26. $Q(x) = x^4 + 10x^2 + 25$

27. $P(x) = x^4 + 3x^2 - 4$

28. $P(x) = x^5 + 7x^3$

29. $P(x) = x^5 + 6x^3 + 9x$

30. $P(x) = x^6 + 16x^3 + 64$

31–40 ■ Find a polynomial with integer coefficients that satisfies the given conditions.

31. P has degree 2, and zeros $1 + i$ and $1 - i$.

32. P has degree 2, and zeros $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$.

33. Q has degree 3, and zeros 3, $2i$, and $-2i$.

34. Q has degree 3, and zeros 0 and i.

35. P has degree 3, and zeros 2 and i.

36. Q has degree 3, and zeros -3 and $1 + i$.

37. R has degree 4, and zeros $1 - 2i$ and 1, with 1 a zero of multiplicity 2.

38. S has degree 4, and zeros $2i$ and $3i$.

39. T has degree 4, zeros i and $1 + i$, and constant term 12.

40. U has degree 5, zeros $\frac{1}{2}$, -1, and $-i$, and leading coefficient 4; the zero -1 has multiplicity 2.

41–58 ■ Find all zeros of the polynomial.

41. $P(x) = x^3 + 2x^2 + 4x + 8$

42. $P(x) = x^3 - 7x^2 + 17x - 15$

43. $P(x) = x^3 - 2x^2 + 2x - 1$

44. $P(x) = x^3 + 7x^2 + 18x + 18$

45. $P(x) = x^3 - 3x^2 + 3x - 2$

46. $P(x) = x^3 - x - 6$

47. $P(x) = 2x^3 + 7x^2 + 12x + 9$

48. $P(x) = 2x^3 - 8x^2 + 9x - 9$

49. $P(x) = x^4 + x^3 + 7x^2 + 9x - 18$

50. $P(x) = x^4 - 2x^3 - 2x^2 - 2x - 3$

51. $P(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$

52. $P(x) = x^5 + x^3 + 8x^2 + 8$ [*Hint:* Factor by grouping.]

53. $P(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$

54. $P(x) = x^4 - x^2 + 2x + 2$

55. $P(x) = 4x^4 + 4x^3 + 5x^2 + 4x + 1$

56. $P(x) = 4x^4 + 2x^3 - 2x^2 - 3x - 1$

57. $P(x) = x^5 - 3x^4 + 12x^3 - 28x^2 + 27x - 9$

58. $P(x) = x^5 - 2x^4 + 2x^3 - 4x^2 + x - 2$

59–64 ■ A polynomial P is given.

(a) Factor P into linear and irreducible quadratic factors with real coefficients.

(b) Factor P completely into linear factors with complex coefficients.

59. $P(x) = x^3 - 5x^2 + 4x - 20$

60. $P(x) = x^3 - 2x - 4$

61. $P(x) = x^4 + 8x^2 - 9$

62. $P(x) = x^4 + 8x^2 + 16$

63. $P(x) = x^6 - 64$

64. $P(x) = x^5 - 16x$

65. By the Zeros Theorem, every nth-degree polynomial equation has exactly n solutions (including possibly some that are repeated). Some of these may be real and some may be imaginary. Use a graphing device to determine how many real and imaginary solutions each equation has.

(a) $x^4 - 2x^3 - 11x^2 + 12x = 0$

(b) $x^4 - 2x^3 - 11x^2 + 12x - 5 = 0$

(c) $x^4 - 2x^3 - 11x^2 + 12x + 40 = 0$

66–68 ■ So far we have worked only with polynomials that have real coefficients. These exercises involve polynomials with real and imaginary coefficients.

66. Find all solutions of the equation.

(a) $2x + 4i = 1$

(b) $x^2 - ix = 0$

(c) $x^2 + 2ix - 1 = 0$

(d) $ix^2 - 2x + i = 0$

67. (a) Show that $2i$ and $1 - i$ are both solutions of the equation

$$x^2 - (1 + i)x + (2 + 2i) = 0$$

but that their complex conjugates $-2i$ and $1 + i$ are not.

(b) Explain why the result of part (a) does not violate the Conjugate Zeros Theorem.

68. (a) Find the polynomial with *real* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and in which the coefficient of the highest power is 1.

(b) Find the polynomial with *complex* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and in which the coefficient of the highest power is 1.

Discovery • Discussion

69. Polynomials of Odd Degree The Conjugate Zeros Theorem says that the complex zeros of a polynomial with real coefficients occur in complex conjugate pairs. Explain how this fact proves that a polynomial with real coefficients and odd degree has at least one real zero.

70. Roots of Unity There are two square roots of 1, namely 1 and -1. These are the solutions of $x^2 = 1$. The fourth roots of 1 are the solutions of the equation $x^4 = 1$ or $x^4 - 1 = 0$. How many fourth roots of 1 are there? Find them. The cube roots of 1 are the solutions of the equation $x^3 = 1$ or $x^3 - 1 = 0$. How many cube roots of 1 are there? Find them. How would you find the sixth roots of 1? How many are there? Make a conjecture about the number of nth roots of 1.

3.6 Rational Functions

A rational function is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We assume that $P(x)$ and $Q(x)$ have no factor in common. Even though rational functions are constructed from polynomials, their graphs look quite different than the graphs of polynomial functions.