WHEN f, f', and f" CHANGE SIGN: TESTS FOR LOCAL EXTREMA & INFLECTION POINTS

When graphing a function f, find where $f(x) = 0$, where $f'(x) = 0$, and where $f''(x) = 0$, if the solutions are easy. Determine where $f'(x)$ is positive and where it is negative. Determine also where $f''(x)$ is positive and where it is negative. The following table contrasts the interpretations of the signs of f, f', and f''. (It is assumed that f, $f^{(1)}$, and $f^{(2)}$ are continuous.)

	Is Positive	*Is Negative*	*Changes Sign*
Where the ordinate $f(x)$	The graph is above the x axis	The graph is below the x axis	The graph crosses the x axis
Where the slope $f'(x)$	The graph slopes upward	The graph slopes downward	The graph has a horizontal tangent and a relative maximum or minimum
Where $f''(x)$	The graph is concave upward (like a cup)	The graph is concave downward	The graph has an inflection point

Keep in mind that the graph can have an inflection point at x_0, even though the second derivative is not defined at x_0 (Example 3). Similarly, a graph can have a maximum or minimum at x_0, even though the first derivative is not defined at x_0. (Consider $f(x) = |x|$ at $x_0 = 0$.)

The second derivative is also useful in searching for relative maxima or minima. For instance, let a be a critical number for the function f and assume that $f''(a)$ happens to be negative. If f'' is continuous in some open interval that contains a, then $f''(a)$ remains negative for a suitably small open interval that contains a. This means that the graph of f is concave downward near $(a, f(a))$, hence lies below its tangent lines. In particular, it lies below the horizontal tangent line at the critical point $(a, f(a))$. Thus the function has a *relative maximum* at the critical number a. This observation suggests the following test for a relative maximum or minimum.

Second-derivative test for local maximum or minimum. Let f be a function with continuous derivative and second derivative. Let a be a critical number for f, that is, $f'(a) = 0$.

If $f''(a) < 0$, f has a local maximum at a.
If $f''(a) > 0$, f has a local minimum at a.

What f ' and f" Tell us About f

Assume f is defined everywhere.

Behavior of f ' ⇒ (implies)	Behavior of f
$f' > 0$ on an interval	f is increasing on the interval
$f' < 0$ on an interval	f is decreasing on the interval
$f'(c) = 0$	f has a horizontal tangent at $x = c$
$f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$ **That is, f' changes from negative to positive at c**	f has a relative minimum at $x = c$
$f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ **That is, f' changes from positive to negative at c**	f has a relative maximum at $x = c$
f ' increasing (or $f''>0$) on an interval	f is concave up on the interval
f ' decreasing (or $f''<0$) on an interval	f is concave down on the interval
f" changes sign at c ALSO f' goes from increasing to decreasing, or vice versa, at c	f has an inflection point at $x=c$

- In proving a relative minimum or maximum, it is <u>never</u> enough to show that the derivative is zero. A single example demonstrates this: Consider $C(x) = x^3$ for $x = 0$. The derivative $C'(x) = 3x^2$ so $C'(0) = 0$ but the function has neither a minimum or maximum there.
- To prove a relative maximum or minimum when $x = c$, it is <u>always</u> necessary to do one of these things:

  Show that f ' (x) <u>changes sign at x=c</u> (i.e use the First Derivative Test)

  Or

  Show that $f'(c)=0$ and $f''(c) \neq 0$ (i.e. use the Second Derivative Test)
- Values of x where $f'(x) = 0$ or $f'(x)$ is undefined are called <u>critical numbers</u>. These are merely <u>candidates</u> for x-values of a maximum or minimum – you must still see if f ' (x) changes sign.
- Even well-labeled sign charts are not enough to show extrema and inflection points. You must <u>state the link</u> between f' and f. See "Lessons Learned at the 2005 Readings."
- If the problem asks for an absolute extreme, you must also evaluate the function at the endpoints of the interval, and compare.