Do Now

True or False

① $\sin(90^\circ - 45^\circ) = \sin 90^\circ - \sin 45^\circ$

$$\sin 45^\circ = \sin 90^\circ - \sin 45^\circ$$

$$\frac{\sqrt{2}}{2} \neq 1 - \frac{\sqrt{2}}{2}$$

False

② $\cos(30^\circ + 60^\circ) = \cos 30^\circ + \cos 60^\circ$

$$\cos 90^\circ = \cos 30^\circ + \cos 60^\circ$$

$$0 \neq \frac{\sqrt{3}}{2} + \frac{1}{2}$$

False

Name:________________________________ Date:____________

PC: Sum and Difference of Angles Formulas Ms. Loughran

Sums and Differences of Angles

Formulas for Sums of Angles	*Formulas for Differences of Angles*
$\sin(x+y) = \sin x \cos y + \cos x \sin y$	$\sin(x-y) = \sin x \cos y - \cos x \sin y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$	$\cos(x-y) = \cos x \cos y + \sin x \sin y$
$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$	$\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

One application of these formulas is to prove other identities.

Examples:

1. Find the exact value of tan 15°.

$$\tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + (\tan 45^\circ)(\tan 30^\circ)}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + (1)\left(\frac{\sqrt{3}}{3}\right)} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$$

$$\frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

2. Prove that the following is an identity: $\tan x + \tan y = \dfrac{\sin(x+y)}{\cos x \cos y}$

$\tan x + \tan y$

$$\frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}$$

$$\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}$$

$$\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}$$

$$\tan x + \tan y$$

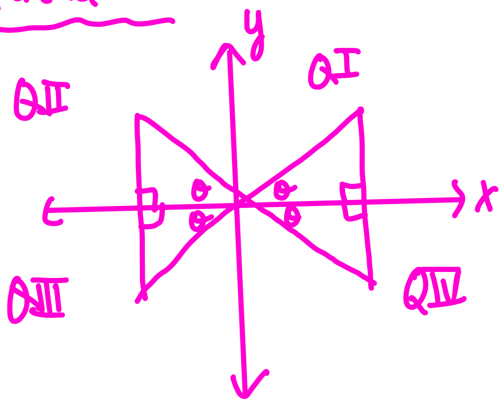
3. Given: $\angle w$ in Quadrant I, $\angle t$ in Quadrant II, $\sin w = \frac{2}{3}$ $\left(\frac{O}{H}\right)$, $\cos t = -\frac{4}{5}$ $\left(\frac{A}{H}\right)$. Find $\sin(w+t)$.

$$\sin(w+t) = \sin w \cos t + \cos w \sin t$$

$$\left(\frac{2}{3}\right)\left(-\frac{4}{5}\right) + \left(\frac{\sqrt{5}}{3}\right)\left(\frac{3}{5}\right) = \frac{-8}{15} + \frac{3\sqrt{5}}{15} = \frac{-8+3\sqrt{5}}{15}$$

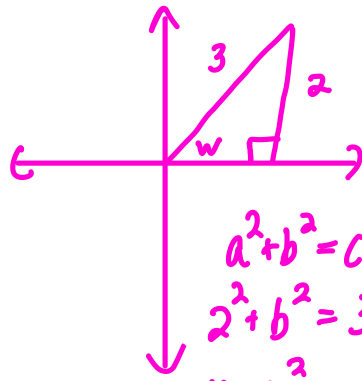
Need cos w and sin t. To find these values we need to use reference Δs.

Reference ∠s



SOH CAH TOA

To find cos w: In QI



$a^2+b^2=c^2$

$2^2+b^2=3^2$

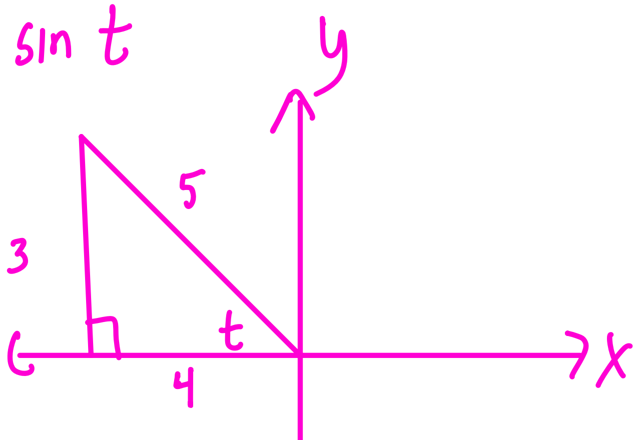
$4+b^2=9$

$b^2=5$

$b=\pm\sqrt{5}$

$\cos w = \frac{\sqrt{5}}{3}$

To find sin t



Pythagorean triple: 3, 4, 5

$4^2+b^2=5^2$

$16+b^2=25$

$b^2=9$

$b=\pm 3$

$\sin t = +\frac{3}{5}$

QI

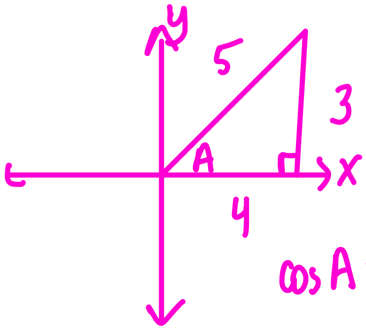
39. If $\sin A = \frac{3}{5}$, $\sin B = \frac{5}{13}$, and angles A and B are acute angles, what is the value of $\cos(A - B)$?

(1) $-\frac{12}{65}$ (2) $\frac{16}{65}$ (3) $\frac{33}{65}$ (4) $\frac{63}{65}$

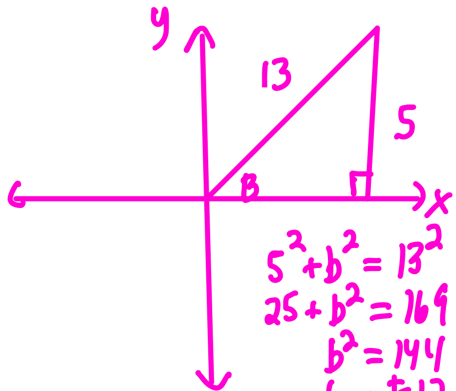
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\left(\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) = \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$$

need cos A and cos B



$\cos A = \pm\frac{4}{5}$



$5^2 + b^2 = 13^2$

$25 + b^2 = 169$

$b^2 = 144$

$b = \pm 12$

Another Pythagorean triple:

5, 12, 13

$\cos B = +\frac{12}{13}$

Exercises

1. Which equation is not a trigonometric identity?
   (1) $\sin^2 x + \cos^2 x = 1$
   (2) $\tan x = \dfrac{\sin x}{\cos x}$
   (3) $\cos(x + y) = \cos x \cos y + \sin x \sin y$
   (4) $\sin(x + y) = \sin x \cos y + \cos x \sin y$

2. $\sin(\theta + 270°)$ is equivalent to
   (1) $\cos\theta$ (2) $2\cos\theta$ (3) $-\cos\theta$ (4) $-\sin\theta$

3. $\sin(180° + A)$ is equivalent to
   (1) $\cos A$ (2) $\sin A$ (3) $-\cos A$ (4) $-\sin A$

4. $\sin(90° - \theta)$ is equivalent to
   (1) $\cos\theta$ (2) $\sin\theta$ (3) $-\cos\theta$ (4) $-\sin\theta$

5. $\cos(\theta + 90°)$ is equivalent to
   (1) $\sin\theta$ (2) $\cos\theta$ (3) $-\sin\theta$ (4) $-\cos\theta$

6. $\cos(2\pi - x)$ is equivalent to
   (1) $-\cos x$ (2) $\cos x$ (3) $-\sin x$ (4) $\sin x$

7. $\tan(x + 45°)$ is equivalent to
   (1) $\dfrac{\tan x - 1}{1 + \tan x}$ (3) $\dfrac{\tan x}{1 + \tan x}$
   (2) $\dfrac{\tan x + 1}{1 - \tan x}$ (4) $\dfrac{\tan x}{1 - \tan x}$

8. $\tan(180° - y)$ is equivalent to
   (1) -1 (3) $-\tan y$
   (2) $\dfrac{-\tan y}{1 + \tan y}$ (4) $\dfrac{1 - \tan y}{1 + \tan y}$

9. $\cos(A - B) - \cos(A + B)$ is equivalent to
   (1) $-2\sin A \sin B$ (3) $2\cos A \cos B$
   (2) $-2\cos B$ (4) $2\sin A \sin B$

10. $\dfrac{\sin(x + y)}{\cos x \cos y}$ is equivalent to
    (1) $1 + \cot x$ (3) $\tan x + \tan y$
    (2) $\tan x + 1$ (4) $\dfrac{1}{\cos y} + \dfrac{1}{\cos x}$

In 11–12, use a sum or difference formula to prove that the given statement is an identity.

11. $\sin(-\theta) = -\sin\theta$ 12. $\tan(-\theta) = -\tan\theta$

In 13–22, prove that the given statement is an identity for all values of the angles for which the expressions are defined.

13. $\sin(x + 45°) = \dfrac{\sqrt{2}}{2}(\sin x + \cos x)$

14. $\cos(60° + y) = \dfrac{1}{2}(\cos y - \sqrt{3}\sin y)$

15. $\tan(45° + x) = \dfrac{1 + \tan x}{1 - \tan x}$

16. $\tan(45° - B) = \dfrac{\cos B - \sin B}{\cos B + \sin B}$

17. $\cos(60° + B) + \cos(60° - B) = \dfrac{1}{\sec B}$

18. $\dfrac{\sin(A - B)}{\sin A \sin B} = \cot B - \cot A$

19. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

20. $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

21. $\dfrac{\cos(x - y)}{\cos(x + y)} = \dfrac{\cot x + \tan y}{\cot x - \tan y}$

22. $\dfrac{\sin(A + B)\cos C}{\sin(A + C)\cos B} = \dfrac{1 + \cot A \tan B}{1 + \cot A \tan C}$

23. a. Using the formula for $\cos(x - y)$, find the exact value of $\cos 15°$ in radical form if $m\angle x = 45°$ and $m\angle y = 30°$.

    b. Using the formula for $\sin(x - y)$, find the exact value of $\sin 15°$ in radical form if $m\angle x = 45°$ and $m\angle y = 30°$.

    c. Find the exact value of $\sin 75°$, using the formula for $\sin(x - y)$ where $m\angle x = 90°$ and $m\angle y = 15°$. Use the values for $\cos 15°$ and $\sin 15°$ found in parts a and b.

24. Since $\cos 75° = \cos(30° + 45°)$, then $\cos 75°$ equals
    (1) $\dfrac{\sqrt{6} - \sqrt{2}}{4}$ (3) $\dfrac{-\sqrt{2} - \sqrt{6}}{4}$
    (2) $\dfrac{-\sqrt{6} + \sqrt{2}}{4}$ (4) $\dfrac{\sqrt{2} + \sqrt{6}}{4}$

25. $\sin 35° \cos 22° + \cos 35° \sin 22°$ equals $= \sin(35° + 22°)$
    (1) $\sin 13°$ (2) $\sin 57°$ (3) $\cos 13°$ (4) $\cos 57°$

26. $\sin 60° \cos 45° - \sin 45° \cos 60°$ equals $\sin(60° - 45°) = \sin 15°$
    $\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$
    $\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$
    (1) 1 (2) 0 (3) $\dfrac{\sqrt{6} - \sqrt{2}}{4}$ (4) $\dfrac{1}{2}$

27. $\cos 70° \cos 40° - \sin 70° \sin 40°$ equals
    (1) $\cos 30°$ (2) $\cos 70°$ (3) $\cos 110°$ (4) $\sin 70°$

28. $\sin 13° \cos 17° + \cos 13° \sin 17°$ equals
    (1) 1 (2) $\dfrac{1}{2}$ (3) $\dfrac{\sqrt{3}}{2}$ (4) 0

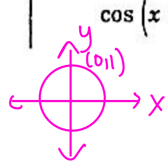
29. $\sin 42° \cos 48° + \cos 42° \sin 48°$ equals
    (1) 1 (2) 0 (3) $\sin 6°$ (4) $\cos 6°$

30. $\sin 96° \cos 24° + \cos 96° \sin 24°$ equals
    (1) $\sin 60°$ (2) $-\sin 60°$ (3) $\cos 60°$ (4) $-\cos 60°$

31. $\sin 210° \cos 30° - \cos 210° \sin 30°$ equals
    (1) 1 (2) -1 (3) 0 (4) 180

32. Express in radical form:
    $\sin 90° \cos 30° - \cos 90° \sin 30°$

33. If $\sin x = \dfrac{3}{5}$ and x is a positive acute angle, find
    $\cos\left(x + \dfrac{\pi}{2}\right)$. $= \cos(x + 90°) = \cos x \cos 90° - \sin x \sin 90°$
    $(0) - \frac{3}{5}(1)$
    $-\frac{3}{5}$



$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$= \left(\frac{3}{5}\right)\left(+\frac{3}{5}\right) + \left(+\frac{4}{5}\right)\left(\frac{4}{5}\right) = \frac{9}{25} + \frac{16}{25} = 1$

34. If A and B are positive acute angles and if $\sin A = \frac{3}{5}$ and $\sin B = \frac{4}{5}$, then $\sin(A + B)$ is equal to

(1) 1 (2) 0 (3) $\frac{7}{5}$ (4) $\frac{24}{25}$

35. If x and y are positive acute angles, and $\sin x = \frac{3}{5}$ and $\sin y = \frac{1}{2}$, then $\cos(x + y)$ is equal to

(1) $\frac{4\sqrt{3}+3}{10}$ (3) $\frac{4}{5} + \frac{\sqrt{3}}{2}$

(2) $\frac{4\sqrt{3}-3}{10}$ (4) $\frac{4}{5} - \frac{\sqrt{3}}{2}$

36. If $\tan x = \frac{1}{2}$ and $\tan y = 1$, the value of $\tan(x + y)$ is

(1) $\frac{1}{2}$ (2) $\frac{3}{4}$ (3) 3 (4) $\frac{3}{2}$

37. If x and y are positive acute angles, and $\sin x = \frac{3}{5}$ and $\sin y = \frac{1}{2}$, then $\sin(x + y)$ is equal to

(1) $\frac{3\sqrt{3}-4}{10}$ (3) $\frac{12}{25} + \frac{\sqrt{3}}{4}$

(2) $\frac{3\sqrt{3}+4}{10}$ (4) $\frac{12}{25} - \frac{\sqrt{3}}{4}$

38. If $\sin \alpha = \frac{3}{5}$, $\tan \beta = \frac{5}{12}$, and α and β are in the first quadrant, then the value of $\cos(\alpha + \beta)$ is

(1) $-\frac{16}{65}$ (2) $\frac{33}{65}$ (3) $\frac{56}{65}$ (4) $\frac{63}{65}$

39. If $\sin A = \frac{3}{5}$, $\sin B = \frac{5}{13}$, and angles A and B are acute angles, what is the value of $\cos(A - B)$?

(1) $-\frac{12}{65}$ (2) $\frac{16}{65}$ (3) $\frac{33}{65}$ (4) $\frac{63}{65}$

40. If $\tan x = \frac{1}{2}$ and $\tan y = \frac{1}{3}$, then the value of $\tan(x + y)$ is

(1) 1 (2) $\frac{5}{7}$ (3) $\frac{1}{5}$ (4) $\frac{1}{7}$

In 41–44, express the answer in simplest form.

41. If $\tan x = 1$ and $\tan y = 2$, find the value of $\tan(x + y)$.

42. If x and y are obtuse angles such that $\sin x = \frac{3}{5}$ and $\sin y = \frac{1}{2}$, find the value of $\sin(x + y)$.

43. If x and y are positive acute angles such that $\cos x = \frac{12}{13}$ and $\cos y = \frac{4}{5}$, find the value of $\cos(x + y)$.

44. If A and B are positive acute angles such that $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$, find the value of $\cos(A + B)$.

Homework 04-02

③ $\sin^2\theta - 2\sin\theta = 3$

$\sin^2\theta - 2\sin\theta - 3 = 0$

$(\sin\theta - 3)(\sin\theta + 1) = 0$

$\sin\theta = 3$ | $\sin\theta = -1$ *unit circle

$\cancel{\theta}$ | $\theta = 270°$

$\{270°\}$

⑤ $2\cos^2\theta = \cos\theta$

$2\cos^2\theta - \cos\theta = 0$

$\cos\theta(2\cos\theta - 1) = 0$

$\cos\theta = 0$ | $\cos\theta = \frac{1}{2}$

$\theta = 90°, 270°$ | ref ∡: $60°$

QI: $60°$

QIV: $300°$

$\{60°, 90°, 270°, 300°\}$

⑦ $\tan\theta(\tan\theta + 1) = \tan\theta + 3$

$\tan^2\theta + \tan\theta = \tan\theta + 3$

$\tan^2\theta = 3$ $\quad \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$

$\tan\theta = \pm\sqrt{3}$

$\tan\theta = \sqrt{3}$ | $\tan\theta = -\sqrt{3}$

ref ∡: $60°$ | ref ∡: $60°$

QI: $60°$ | QII $120°$

QIII $240°$ | QIV $300°$

⑨ $\frac{\cos\theta}{1} = \frac{1}{\cos\theta}$

$\cos^2\theta = 1$

$\cos\theta = \pm 1$

$\cos\theta = 1$ $\quad$ $\cos\theta = -1$

*unit circle

$0°, 360°$ $\quad$ $180°$

$\{0, \pi, 2\pi\}$

⑫ $\sec^2\beta = 6\sec\beta + 7$

$\sec^2\beta - 6\sec\beta - 7 = 0$

$(\sec\beta - 7)(\sec\beta + 1) = 0$

$\sec\beta = 7$ | $\sec\beta = -1$

$\cos\beta = \frac{1}{7}$ | $\cos\beta = -1$ ← unit circle

$\cos^{-1}\left(\frac{1}{7}\right) = 81.786..$ | $\beta = 180°$

QI $\beta = 81.8°$

QIV $\beta = 360° - 81.8° = 278.2°$

$\{81.8°, 180.0°, 278.2°\}$

(13) $3\sin^2\beta+\sin\beta+5=4(1-\sin\beta)$

$3\sin^2\beta+\sin\beta+5=4-4\sin\beta$

$3\sin^2\beta+5\sin\beta+1=0$

$$\sin\beta=\frac{-5\pm\sqrt{(5)^2-4(3)(1)}}{2(3)}=\frac{-5\pm\sqrt{13}}{6}$$

$$\sin\beta=\frac{-5+\sqrt{13}}{6}=-.232\ldots \quad \sin\beta=\frac{-5-\sqrt{13}}{6}\approx-1.434$$

$\emptyset$

$$\sin^{-1}\left(-\left(\frac{-5+\sqrt{13}}{6}\right)\right)=13.43\ldots$$

$\{193.4°, 346.6°\}$

make this value (+) to find ref ∡

QIII $\theta=180°+13.4°=193.4°$

QIV $\theta=360°-13.4°=346.6°$

(16) $3\tan^2\beta-5\tan\beta=2$

$3\tan^2\beta-5\tan\beta-2=0$ $\quad ac=-6 \quad b=-5$

$3\tan^2\beta-6\tan\beta+\tan\beta-2=0$

$3\tan\beta(\tan\beta-2)+1(\tan\beta-2)=0$

$\{63.4°, 161.6°, 243.4°, 341.6°\}$

$(3\tan\beta+1)(\tan\beta-2)=0$

$\tan\beta=\frac{-1}{3}$ $\qquad \tan\beta=2$

$\tan^{-1}(2)=63.434\ldots$

find ref ∡ $\tan^{-1}\left(+\frac{1}{3}\right)=18.434\ldots$ QI $\beta=63.4°$

QII $\beta=180°-18.4°=161.6°$ QIII $\beta=180°+63.4°=243.4°$

QIV $\beta=360°-18.4°=341.6°$

(15) $3(1-\sin^2\beta) = \sin\beta$

$3 - 3\sin^2\beta = \sin\beta$

$0 = 3\sin^2\beta + \sin\beta - 3$

$\sin\beta = \dfrac{-1 \pm \sqrt{1^2 - 4(3)(-3)}}{2(3)}$

$\sin\beta = \dfrac{1 \pm \sqrt{37}}{6}$

$\sin\beta = \dfrac{1+\sqrt{37}}{6}$ $\quad$ $\sin\beta = \dfrac{1-\sqrt{37}}{6}$

$\sin\beta = 1.180...$ $\quad$ $\sin\beta = -.8471...$

$\emptyset$

ref ∡: $\sin^{-1}(+.8471..) = 57.900...^\circ$

Q III : $180 + 57.900..^\circ = 237.900...$

Q IV $\quad 360 - 57.900..^\circ = 302.099..^\circ$

$\{57.9^\circ, 302.1^\circ\}$

(17) $2\sin^2\beta = \sin\beta$

$2\sin^2\beta - \sin\beta = 0$

$\sin\beta(2\sin\beta - 1) = 0$

$\sin\beta = 0$ | $\sin\beta = \frac{1}{2}$

$0^\circ, 180^\circ, 360^\circ$

ref∡: 30°

QI 30°

QII 150°

* the interval was $180^\circ \le \beta < 270^\circ$

so $\{180^\circ\}$

(14) $6\cos^2\beta + 6\cos\beta + 2 = 1 + \cos\beta$

$6\cos^2\beta + 5\cos\beta + 1 = 0$

$6\cos^2\beta + 3\cos\beta + 2\cos\beta + 1 = 0$

$3\cos\beta(2\cos\beta + 1) + 1(2\cos\beta + 1) = 0$

$(3\cos\beta + 1)(2\cos\beta + 1) = 0$ $\quad$ $\{109.5^\circ, 250.5^\circ, 120.0^\circ, 240.0^\circ\}$

$\cos\beta = -\frac{1}{3}$ | $\cos\beta = -\frac{1}{2}$

$\cos^{-1}(+\frac{1}{3}) = 70.528$ ← find ref ∡ (remember ignore ⊖) $\cos^{-1}(+\frac{1}{2}) = 60^\circ$

$\approx 70.5^\circ$ $\quad$ since $\cos\beta$ ⊖ angles must be in QII + QIII

QII $\beta = 180^\circ - 70.5 = 109.5^\circ$ $\quad$ QII $\beta = 180^\circ - 60^\circ = 120^\circ$

QIII $\beta = 180^\circ + 70.5^\circ = 250.5^\circ$ $\quad$ QIII $\beta = 180^\circ + 60^\circ = 240^\circ$