Name:________________________________ Date:__________

PCH: Using Matrices to Solve Systems of Linear Equations Ms. Loughran

We can use matrices as a streamlined technique for solving systems of linear equations.

Model:

1. Given:
$$\begin{aligned} x-2y+3z &= 9 \\ -x+3y \quad\quad &= -4 \\ 2x-5y+5z &= 17 \end{aligned}$$

Coefficient Matrix 3x3

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{bmatrix}$$

Augmented Matrix 3x4

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

***constant terms are not included**

***constant terms are included**

To solve a linear system of equations we will use an augmented matrix.

To solve a matrix we use the elementary row operations that we discussed. Remember the 3 elementary row operations are the same three operations that we used to solve the linear systems of equations by elimination.

- interchange rows
- multiply a row by a nonzero constant
- add a multiple of one row to another row

* Row echelon form

$$\left[\begin{array}{ccc|c} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \end{array}\right]$$

Reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right]$$

Let's get back to solving the system.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \quad R_1 + R_2 \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$-2R_1 + R_3$

$-2R_1 \; [-2 \;\; 4 \;\; -6 \;\; -18]$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \quad R_2 + R_3 \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\tfrac{1}{2}R_3 \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$z = 2$

$y + 3(2) = 5$

$y + 6 = 5$

$y = -1$

$x - 2(-1) + 3(2) = 9$

$x + 8 = 9$

$x = 1$

$(1, -1, 2)$

This last matrix is said to be in row-echelon form. The term echelon refers to the stair step pattern formed by the nonzero elements of the matrix. To be in row echelon form, a matrix must have these properties:

1. **All rows consisting entirely of zeros occur at the bottom of the matrix.**

2. **For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1).**

3. **For two successive (nonzero) rows, the leading 1 in the higher row is father to the left than the leading 1 in the lower row.**

A matrix in row-echelon form is in *reduced row-echelon form* if every column that has a leading 1 has zeros in every position above and below its leading one.

2. Solve the following system using matrices:

$$\begin{aligned} x + y - 5z &= 3 \\ x \quad - 2z &= 1 \\ 2x - y - z &= 0 \end{aligned}$$

$-R_1 \; [-1 \; -1 \; 5 \mid -3]$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$-2R_1 \; [-2 \; -2 \; 10 \; -6]$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & -1 & 0 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{array}\right]$$

$-3R_2 \; [0 \; 3 \; -9 \; 6]$

$$\xrightarrow{3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

infinitely many solutions

$(2z+1, 3z+2, z)$

$x - 2z = 1$

$x = 2z + 1$

$2z + 1 + y - 5z = 3$

$y - 3z = 2$

$y = 3z + 2$

3. Solve the following system using matrices:

$$\begin{aligned} x-2y+z&=7\\ 3x+y-z&=2\\ 2x+3y+2z&=7 \end{aligned}$$

$-3R_1\ [-3\quad 6\quad -3\quad -21]$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 7\\ 3 & 1 & -1 & 2\\ 2 & 3 & 2 & 7 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 7\\ 0 & 7 & -4 & -19\\ 2 & 3 & 2 & 7 \end{array}\right]$$

$-2R_1\ [-2\quad 4\quad -2\quad -14]$

$$\xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 7\\ 0 & 7 & -4 & -19\\ 0 & 7 & 0 & -7 \end{array}\right] \xrightarrow{\frac{1}{7}R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 7\\ 0 & 1 & 0 & -1\\ 0 & 7 & -4 & -19 \end{array}\right]$$

$-7R_2\ [0\quad -7\quad 0\quad 7]$

$$\xrightarrow{-7R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 7\\ 0 & 1 & 0 & -1\\ 0 & 0 & -4 & -12 \end{array}\right] \xrightarrow{-\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 7\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 3 \end{array}\right]$$

Steps:

1. convert system of equations into an augmented matrix
2. put the matrix in row echelon form
3. find the value of z from the last row, back substitute to find y and then x.

$(2, -1, 3)$

$z = 3$

$y = -1$

$x - 2(-1) + 3 = 7$

$x = 2$