Name: ____ Date: _____________

PCH: Unit Circle Ms. Loughran

Do Now:

1. The half life of radium-226 is 1590 years. If the initial mass is 150 mg, Find:
   (a) the mass that will remain after 1000 years.
   (b) after how many years will only 50 mg remain?

$r(t) = 150e^{-\frac{\ln 2}{1590}t}$

(a) $r(1000) = 150e^{-\frac{\ln 2}{1590}(1{,}000)}$

$96.997\ldots \approx 97$ mgs

(b) $50 = 150e^{-\frac{\ln 2}{1590}t}$

$\frac{1}{3} = e^{-\frac{\ln 2}{1590}t}$

$\ln\frac{1}{3} = -\frac{\ln 2}{1590}t$

$t = \ln\left(\frac{1}{3}\right)\cdot\frac{1590}{-\ln 2}$

$t = 2{,}520.090\ldots \approx 2{,}520$ yrs

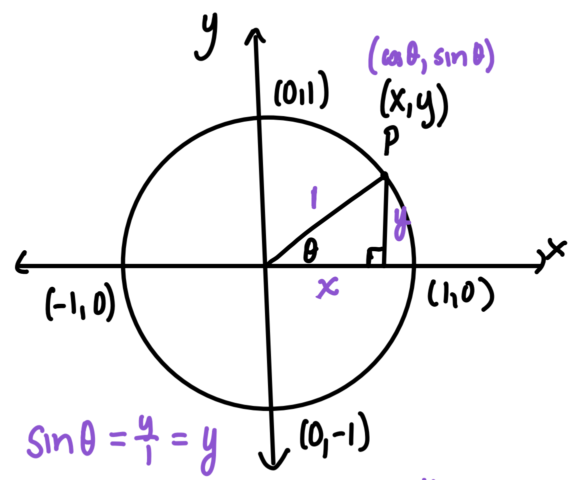
The unit circle is a circle with center at the origin and radius 1.
Therefore its equation is: $x^2 + y^2 = 1$
The first two trigonometric functions we will study are sine and cosine.

In the figure at the right, angle θ is in standard position. Point P represents the intersection of the unit circle and the terminal side of angle θ in standard position. We define the functions as follows:

The sine of θ is the y-coordinate of P.
The cosine of θ is the x-coordinate of P.

Also we can express tangent in terms of sine and cosine.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x},\ \neq 0$$



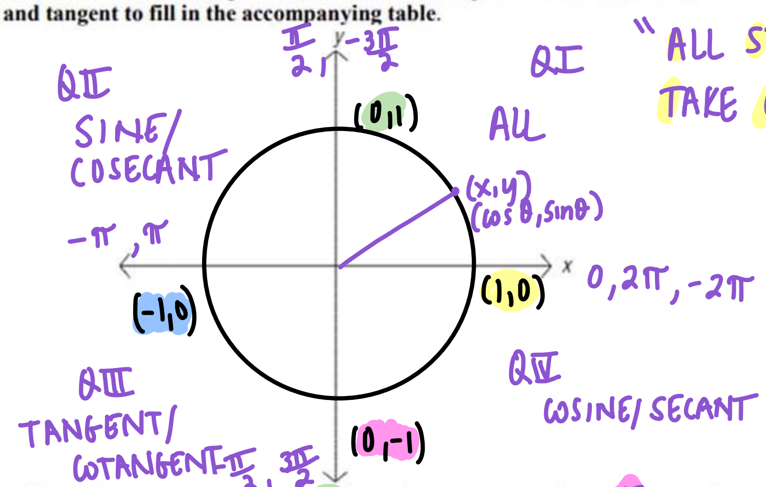
$\sin\theta = \frac{y}{1} = y$

$\cos\theta = \frac{x}{1} = x$

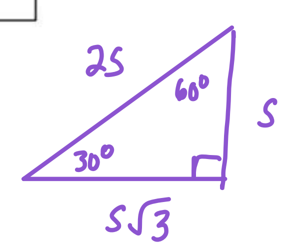
$\tan\theta = \frac{y}{x}$

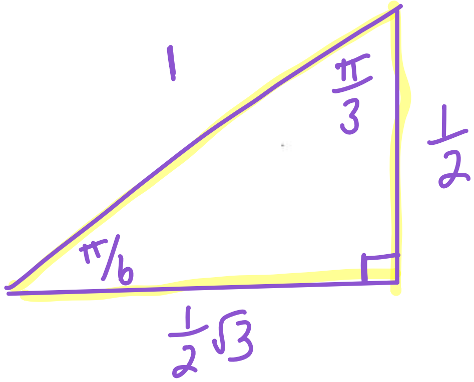
Notice the signs of these functions depend on the quadrant in which angle θ lies.
Draw the unit circle on the axes provided. Label the four points where the circle intersects the axes. Use those points and what we have just learned about sine, cosine and tangent to fill in the accompanying table.

"ALL STUDENTS TAKE CALCULUS"



θ in radians	0	$\pi/2$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	1	0	-1	0
$\cos\theta$	1	0	-1	0	1
$\tan\theta$	0	undefined	0	undefined	0





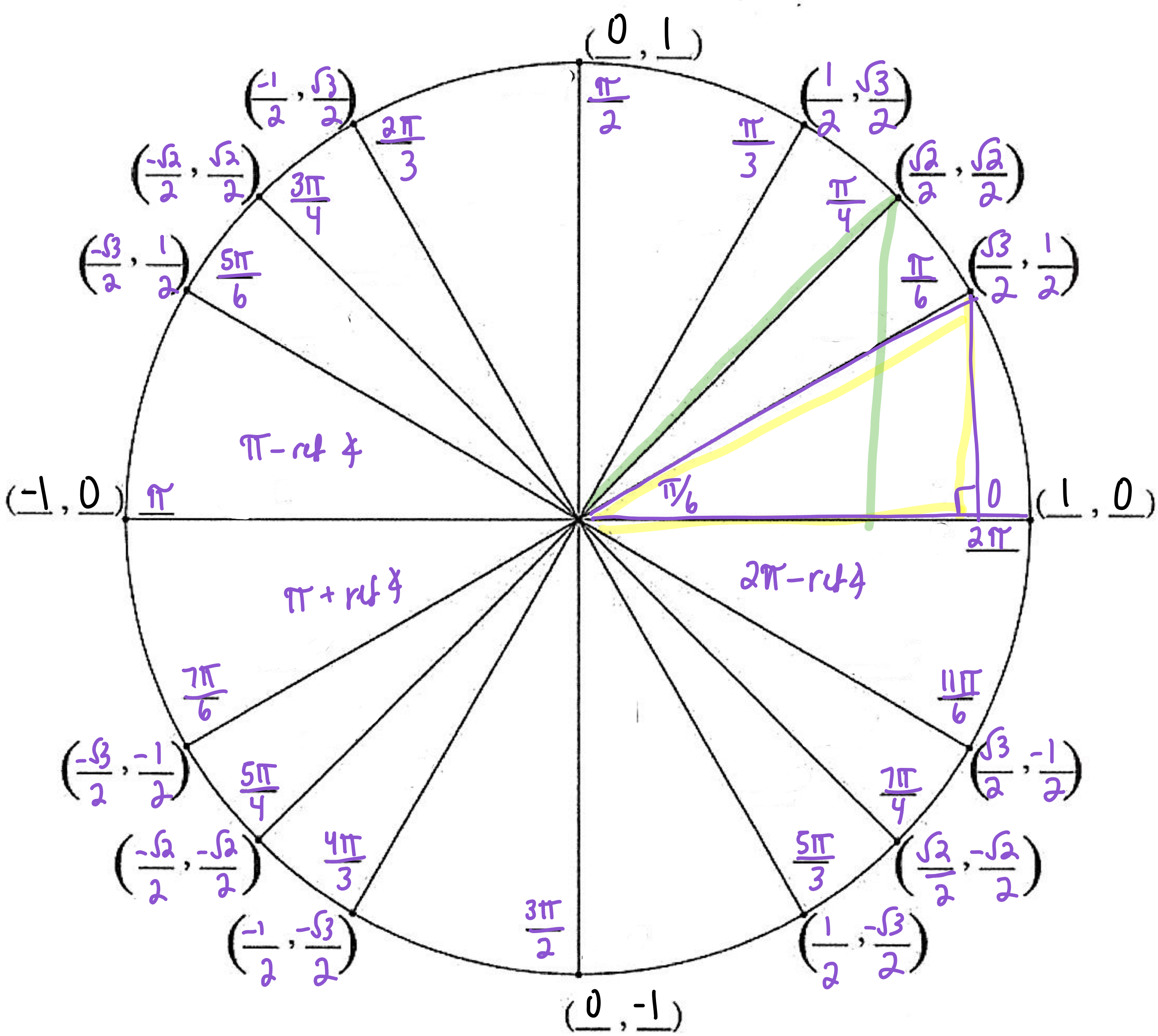
$\sin\frac{\pi}{6} = \frac{1}{2}$

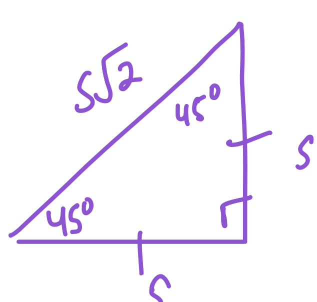
$\cos\frac{\pi}{6} = \sqrt{3}/2$

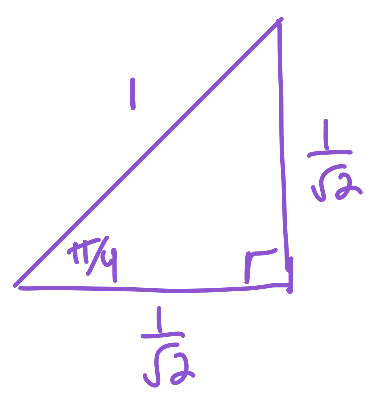
THE UNIT CIRCLE

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$







$s\sqrt{2} = 1$

$s = \frac{1}{\sqrt{2}}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos\frac{\pi}{4} = \sqrt{2}/2$

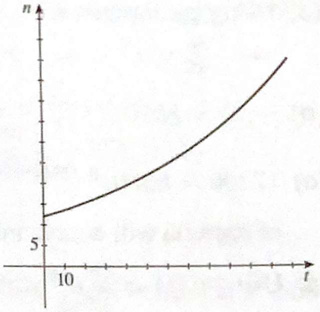
Homework 04-02

2. (a) The relative growth rate is $0.012 = 1.2\%$.

(b) $n(5) = 12e^{0.012(5)} = 12e^{0.06} \approx 12.74$ million fish.

(c) $30 = 12e^{0.012t} \quad \Leftrightarrow \quad 2.5 = e^{0.012t} \quad \Leftrightarrow \quad 0.012t = \ln 2.5 \quad \Leftrightarrow$ $t = \dfrac{\ln 2.5}{0.012} \approx 76.36$. Thus the fish population reaches 30 million after about 76 years.

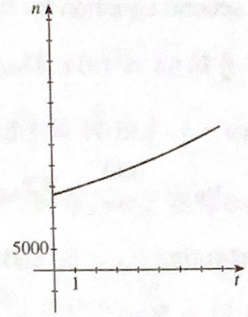
(d)



3. (a) $r = 0.08$ and $n(0) = 18000$. Thus the population is given by the formula $n(t) = 18{,}000e^{0.08t}$.

(b) $t = 2008 - 2000 = 8$. Then we have $n(8) = 18000e^{0.08(8)} = 18000e^{0.64} \approx 34{,}137$. Thus there should be 34,137 foxes in the region by the year 2008.

(c)



4. $n(t) = n_0e^{rt}$; $n_0 = 110$ million, $t = 2020 - 1995 = 25$

(a) $r = 0.03$; $n(25) = 110{,}000{,}000e^{0.03(25)} = 110{,}000{,}000e^{0.75} \approx 232{,}870{,}000$. Thus at a 3% growth rate, the projected population will be approximately 233 million people by the year 2020.

(b) $r = 0.02$; $n(25) = 110{,}000{,}000e^{0.02(25)} = 110{,}000{,}000e^{0.50} \approx 181{,}359{,}340$. Thus at a 2% growth rate, the projected population will be approximately 181 million people by the year 2020.

5. (a) $n(t) = 112{,}000e^{0.04t}$.

(b) $t = 2000 - 1994 = 6$ and $n(6) = 112{,}000e^{0.04(6)} \approx 142380$. The projected population is about142,000.

(c) $200{,}000 = 112{,}000e^{0.04t} \quad \Leftrightarrow \quad \frac{25}{14} = e^{0.04t} \quad \Leftrightarrow \quad 0.04t = \ln\left(\frac{25}{14}\right) \quad \Leftrightarrow \quad t = 25\ln\left(\frac{25}{14}\right) \approx 14.5$. Since $1994 + 14.5 = 2008.5$, the population will reach 200,000 during the year 2008.

6. (a) $n(t) = n_0e^{rt}$ with $n_0 = 85$ and $r = 0.18$. Thus $n(t) = 85e^{0.18t}$.

(b) $n(3) = 85e^{0.18(3)} \approx 146$ frogs.

(c) $600 = 85e^{0.18t} \quad \Leftrightarrow \quad \frac{120}{17} = e^{0.18t} \quad \Leftrightarrow \quad 0.18t = \ln\left(\frac{120}{17}\right) \quad \Leftrightarrow \quad t = \frac{1}{0.18}\ln\left(\frac{120}{17}\right) \approx 10.86$. So the population will reach 600 frogs in about 11 years.

7. (a) The deer population in 1996 was 20,000.

(b) Using the model $n(t) = 20{,}000e^{rt}$ and the point $(4, 31000)$, we have $31{,}000 = 20{,}000e^{4r} \quad \Leftrightarrow \quad 1.55 = e^{4r} \quad \Leftrightarrow$ $4r = \ln 1.55 \quad \Leftrightarrow \quad r = \frac{1}{4}\ln 1.55 \approx 0.1096$. Thus $n(t) = 20{,}000e^{0.1096t}$

(c) $n(8) = 20{,}000e^{0.1096(8)} \approx 48{,}218$, so the projected deer population in 2004 is about 48,000.

(d) $100{,}000 = 20{,}000e^{0.1096t} \quad \Leftrightarrow \quad 5 = e^{0.1096t} \quad \Leftrightarrow \quad 0.1096t = \ln 5 \quad \Leftrightarrow \quad t = \dfrac{\ln 5}{0.1096} \approx 14.63$. Since $1996 + 14.63 = 2010.63$, the deer population will reach 100,000 during the year 2010.

8. (a) Since the population grows exponentially, the population is represented by $n(t) = n_0e^{rt}$, with $n_0 = 1500$ and $n(30) = 3000$. Solving for r, we have $3000 = 1500e^{30r} \quad \Leftrightarrow \quad 2 = e^{30r} \quad \Leftrightarrow \quad 30r = \ln 2 \quad \Leftrightarrow$ $r = (\ln 2)/30 \approx 0.023$. Thus $n(t) = 1500e^{0.023t}$.

... is $n(120) = 1500e^{0.023(120)} \approx 24{,}000$.

(b) $n(2) = 8600e^{0.1508(2)} \approx 11627$. Thus the number of bacteria after 2 hours is about 11,600.

(c) $17200 = 8600e^{0.1508t} \Leftrightarrow 2 = e^{0.1508t} \Leftrightarrow 0.1508t = \ln 2 \Leftrightarrow t = \dfrac{\ln 2}{0.1508} \approx 4.596$. Thus the number of bacteria will double in about 4.6 hours.

10. (a) Using $n(t) = n_0e^{rt}$ with $n(2) = 400$ and $n(6) = 25{,}600$, we have $n_0e^{2r} = 400$ and $n_0e^{6r} = 25{,}600$. Dividing the second equation by the first gives $\dfrac{n_0e^{6r}}{n_0e^{2r}} = \dfrac{25{,}600}{400} = 64 \Leftrightarrow e^{4r} = 64 \Leftrightarrow 4r = \ln 64 \Leftrightarrow$ $r = \frac{1}{4}\ln 64 \approx 1.04$. Thus the relative rate of growth is about 104%.

(b) Since $r = \frac{1}{4}\ln 64 = \frac{1}{2}\ln 8$, we have from part (a) $n(t) = n_0e^{\left(\frac{1}{2}\ln 8\right)t}$. Since $n(2) = 400$, we have $400 = n_0e^{\ln 8}$ $\Leftrightarrow n_0 = \dfrac{400}{e^{\ln 8}} = \frac{400}{8} = 50$. So the initial size of the culture was 50.

(c) Substituting $n_0 = 50$ and $r = 1.04$, we have $n(t) = n_0e^{rt} = 50e^{1.04t}$.

(d) $n(4.5) = 50e^{1.04(4.5)} = 50e^{4.68} \approx 5388.5$, so the size after 4.5 hours is approximately 5400.

(e) $n(t) = 50{,}000 = 50e^{1.04t} \Leftrightarrow e^{1.04t} = 1000 \Leftrightarrow 1.04t = \ln 1000 \Leftrightarrow t = \dfrac{\ln 1000}{1.04} \approx 6.64$. Hence the population will reach 50,000 after roughly $6\frac{2}{3}$ hours.

11. (a) $2n_0 = n_0e^{0.02t} \Leftrightarrow 2 = e^{0.02t} \Leftrightarrow 0.02t = \ln 2 \Leftrightarrow t = 50\ln 2 \approx 34.65$. So we have $t = 1995 + 34.65 = 2029.65$, and hence at the current growth rate the population will double by the year 2029.

(b) $3n_0 = n_0e^{0.02t} \Leftrightarrow 3 = e^{0.02t} \Leftrightarrow 0.02t = \ln 3 \Leftrightarrow t = 50\ln 3 \approx 54.93$. So we have $t = 1995 + 54.93 = 2049.93$, and hence at the current growth rate the population will triple by the year 2050.

12. (a) Calculating dates relative to 1950 gives $n_0 = 10{,}586{,}223$ and $n(30) = 23{,}668{,}562$. Then $n(30) = 10{,}586{,}223e^{30r} = 23{,}668{,}562 \Leftrightarrow e^{30r} = \frac{23{,}668{,}562}{10{,}586{,}223} \approx 2.2358 \Leftrightarrow 30r = \ln 2.2358 \Leftrightarrow$ $r = \frac{1}{30}\ln 2.2358 \approx 0.0268$. Thus $n(t) = 10{,}586{,}223e^{0.0268t}$.

(b) $2(10{,}586{,}223) = 10{,}586{,}223e^{0.0268t} \Leftrightarrow 2 = e^{0.0268t} \Leftrightarrow \ln 2 = 0.0268t \Leftrightarrow t = \dfrac{\ln 2}{0.0268} \approx 25.86$. So the population doubles in about 26 years.

(c) $t = 2000 - 1950 = 50$; $n(50) \approx 10586223e^{0.0268(50)} \approx 40{,}429{,}246$ and so the population in the year 2000 will be approximately 40,429,000.

13. $n(t) = n_0e^{2t}$. When $n_0 = 1$, the critical level is $n(24) = e^{2(24)} = e^{48}$. We solve the equation $e^{48} = n_0e^{2t}$, where $n_0 = 10$. This gives $e^{48} = 10e^{2t} \Leftrightarrow 48 = \ln 10 + 2t \Leftrightarrow 2t = 48 - \ln 10 \Leftrightarrow$ $t = \frac{1}{2}(48 - \ln 10) \approx 22.85$ hours.

14. From the formula for radioactive decay, we have $m(t) = m_0e^{-rt}$, where $r = \dfrac{\ln 2}{h}$.

(a) We have $m_0 = 22$ and $h = 1600$, so $r = \dfrac{\ln 2}{1600} \approx 0.000433$ and the amount after t years is given by $m(t) = 22e^{-0.000433t}$.

(b) $m(4000) = 22e^{-0.000433(4000)} \approx 3.89$, so the amount after 4000 years is about 4 mg.

(c) We have to solve for t in the equation $18 = 22\,e^{-0.000433t}$. This gives $18 = 22e^{-0.000433t} \Leftrightarrow \frac{9}{11} = e^{-0.000433t}$ $\Leftrightarrow -0.000433t = \ln\left(\frac{9}{11}\right) \Leftrightarrow t = \dfrac{\ln\left(\frac{9}{11}\right)}{-0.000433} \approx 463.4$, so it takes about 463 years.

15. (a) Using $m(t) = m_0 e^{-rt}$ with $m_0 = 10$ and $h = 30$, we have $r = \frac{\ln 2}{h} = \frac{\ln 2}{30} \approx 0.0231$. Thus $m(t) = 10e^{-0.0231t}$.

(b) $m(80) = 10e^{-0.0231(80)} \approx 1.6$ grams.

(c) $2 = 10e^{-0.0231t} \quad \Leftrightarrow \quad \frac{1}{5} = e^{-0.0231t} \quad \Leftrightarrow \quad \ln\left(\frac{1}{5}\right) = -0.0231t \quad \Leftrightarrow \quad t = \frac{-\ln 5}{-0.0231} \approx 70$ years.

16. (a) $m(60) = 40e^{-0.0277(60)} \approx 7.59$, so the mass remaining after 60 days is about 8 g.

(b) $10 = 40e^{-0.0277t} \quad \Leftrightarrow \quad 0.25 = e^{-0.0277t} \quad \Leftrightarrow \quad \ln 0.25 = -0.0277t \quad \Leftrightarrow \quad t = -\frac{\ln 0.25}{0.0277} \approx 50.05$, so it takes about 50 days.

(c) We need to solve for t in the equation $20 = 40e^{-0.0277t}$. We have $20 = 40e^{-0.0277t} \quad \Leftrightarrow \quad e^{-0.277t} = \frac{1}{2} \quad \Leftrightarrow \quad -0.0277t = \ln\frac{1}{2} \quad \Leftrightarrow \quad t = \frac{\ln\frac{1}{2}}{-0.0277} \approx 25.02$. Thus the half-life of thorium-234 is about 25 days.

17. By the formula in the text, $m(t) = m_0 e^{-rt}$ where $r = \frac{\ln 2}{h}$, so $m(t) = 50e^{-[(\ln 2)/28]t}$. We need to solve for t in the equation $32 = 50e^{-[(\ln 2)/28]t}$. This gives $e^{-[(\ln 2)/28]t} = \frac{32}{50} \quad \Leftrightarrow \quad -\frac{\ln 2}{28}t = \ln\left(\frac{32}{50}\right) \quad \Leftrightarrow \quad t = -\frac{28}{\ln 2} \cdot \ln\left(\frac{32}{50}\right) \approx 18.03$, so it takes about 18 years.

18. From the formula for radioactive decay, we have $m(t) = m_0 e^{-rt}$, where $r = \frac{\ln 2}{h}$. Since $h = 30$, we have $r = \frac{\ln 2}{30} \approx 0.0231$ and $m(t) = m_0 e^{-0.0231t}$. In this exercise we have to solve for t in the equation $0.05m_0 = m_0 e^{-0.0231t}$ $\Leftrightarrow \quad e^{-0.0231t} = 0.05 \quad \Leftrightarrow \quad -0.0231t = \ln 0.05 \quad \Leftrightarrow \quad t = \frac{\ln 0.05}{-0.0231} \approx 129.7$. So it will take about 130 s.

19. By the formula for radioactive decay, we have $m(t) = m_0 e^{-rt}$, where $r = \frac{\ln 2}{h}$, in other words $m(t) = m_0 e^{-[(\ln 2)/h]t}$. In this exercise we have to solve for h in the equation $200 = 250e^{-[(\ln 2)/h]\cdot 48} \quad \Leftrightarrow \quad 0.8 = e^{-[(\ln 2)/h]\cdot 48} \quad \Leftrightarrow \quad \ln(0.8) = -\frac{\ln 2}{h} \cdot 48 \quad \Leftrightarrow \quad h = -\frac{\ln 2}{\ln 0.8} \cdot 48 \approx 149.1$ hours. So the half-life is approximately 149 hours.

20. From the formula for radioactive decay, we have $m(t) = m_0 e^{-rt}$, where $r = \frac{\ln 2}{h}$. In other words, $m(t) = m_0 e^{-[(\ln 2)/h]t}$.

(a) Using $m(3) = 0.58m_0$, we have to solve for h in the equation $0.58m_0 = m(3) = m_0 e^{-[(\ln 2)/h]3}$. Then $0.58m_0 = m_0 e^{-[(3\ln 2)/h]} \quad \Leftrightarrow \quad e^{-[(3\ln 2)/h]} = 0.58 \quad \Leftrightarrow \quad -\frac{3\ln 2}{h} = \ln 0.58 \quad \Leftrightarrow \quad h = -\frac{3\ln 2}{\ln 0.58} \approx 3.82$ days. Thus the half-life of Radon-222 is about 3.82 days.

(b) Here we have to solve for t in the equation $0.2m_0 = m_0 e^{-[(\ln 2)/3.82]t}$. So we have $0.2m_0 = m_0 e^{-[(\ln 2)/3.82]t} \quad \Leftrightarrow \quad 0.2 = e^{-[(\ln 2)/3.82]t} \quad \Leftrightarrow \quad -\frac{\ln 2}{3.82}t = \ln 0.2 \quad \Leftrightarrow \quad t = -\frac{3.82\ln 0.2}{\ln 2} \approx 8.87$. So it takes roughly 9 days for a sample of Radon-222 to decay to 20% of its original mass.

21. By the formula in the text, $m(t) = m_0 e^{-[(\ln 2)/h]\cdot t}$, so we have $0.65 = 1 \cdot e^{-[(\ln 2)/5730]\cdot t} \quad \Leftrightarrow \quad \ln(0.65) = -\frac{\ln 2}{5730}t$ $\Leftrightarrow \quad t = -\frac{5730\ln 0.65}{\ln 2} \approx 3561$. Thus the artifact is about 3560 years old.

22. From the formula for radioactive decay, we have $m(t) = m_0 e^{-rt}$ where $r = \frac{\ln 2}{h}$. Since $h = 5730$, $r = \frac{\ln 2}{5730} \approx 0.000121$ ... $0.59m_0 = m_0 e^{-0.000121t} \quad \Leftrightarrow \quad e^{-0.000121t} = 0.59$