Name: ________________________________  Date: _________

PCH: More Polys  Ms. Loughran

Do Now: Also the first 5 examples on the sheet that says Factor Theorem

1. Divide $P(x) = 2x^5 + 4x^4 - 4x^3 - x - 3$ by $x^2 - 2$

$$
\begin{array}{r|l}
 & 2x^3 + 4x^2 + 8 \\
x^2 - 2 & 2x^5 + 4x^4 - 4x^3 \quad - x - 3 \\
 & 2x^5 \quad - 4x^3 \\
\hline
 & 4x^4 \quad - x - 3 \\
 & 4x^4 - 8x^2 \\
\hline
 & 8x^2 - x - 3 \\
 & 8x^2 \quad - 16 \\
\hline
 & -x + 13
\end{array}
$$

$-x + 13$ ← remainder

$2x^3 + 4x^2 + 8 + \frac{-x+13}{x^2-2}$

$2x^3 + 4x^2 + 8 - \frac{x-13}{x^2-2}$

remainder

$$
\begin{array}{c|cccc|cc}
 & 2 & 4 & -4 & 0 & -1 & -3 \\
0 & & 0 & 0 & 0 & 0 & \\
2 & & & 4 & 8 & 0 & 16 \\
\hline
 & 2 & 4 & 0 & 8 & -1 & 13
\end{array}
$$

quotient $2x^3 + 4x^2 + 8$

remainder: $-x + 13$

# THE FACTOR THEOREM

L. Lewis

**THEOREM:** $f(x)$ is a polynomial function of degree $\geq 1$ and $x - c$ is a factor of $f$ if and only if $f(c) = 0$ (i.e. $c$ is a zero of $f$).

**EXAMPLE:** If $f(x) = 2x - 6$ and 3 is a zero of $f(x)$, then $X-3$ is a factor of $f(x)$.

**EXAMPLE:** If $x + 1$ is a factor of $f(x)$, then $-1$ is a zero of $f(x)$.

**EXAMPLE:** If $f(x) = (x - 3)(x + 4)(2x - 1)$, then the zeros of $f(x)$ are: $3, -4, \frac{1}{2}$.

**EXAMPLE:** If $f(4) = 0$, then a factor of $f$ is: $X-4$.

**EXAMPLE:** If $3x - 1$ is a factor of $f(x)$, then $f(\frac{1}{3}) = 0$.

**Note: The complete factorization of a polynomial will include factors with ONLY integral coefficients.**

## PROBLEMS

1. Show 2 ways that $y - 1$ is a factor of $y^3 - 3y^2 + 3y - 1$.

plug in 1 for y and show it = 0

$1^3 - 3(1)^2 + 3(1) - 1 = 1 - 3 + 3 - 1 = 0$

or

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

← show remainder is 0

or

Long Division to show the remainder is 0

3. Factors of $x^3 + x^2 - 4x - 4$ are $(x - 2)$, $(x + 2)$ and $(x + 1)$. What are the zeros of the polynomial?

$\{\pm 2, -1\}$

5. If $f(x) = x^3 - x^2 + kx - 12$, find $k$ so that $f(x)$ is exactly divisible by $x + 3$.

$f(-3) = 0$

$(-3)^3 - (-3)^2 + k(-3) - 12 = 0$

$-3k - 48 = 0$

$k = -16$

$$\begin{array}{r|rrrr} -3 & 1 & -1 & k & -12 \\ & & -3 & 12 & -3k-36 \\ \hline & 1 & -4 & k+12 & 0 \end{array}$$

$-12 - 3k - 36 = 0$

$-3k = 48$

$k = -16$

7. If $f(x) = ax^5 + ax^4 + 13x^3 - 11x^2 - 10x - a$, find $f(1)$ if $f(-1) = 0$.

need a

$a(-1)^5 + a(-1)^4 + 13(-1)^3 - 11(-1)^2 - 10(-1) - a = 0$

$\cancel{-a} + \cancel{a} - 13 - 11 + 10 - a = 0$

$-14 - a = 0$

$-14 = a$

$f(x) = -14x^5 - 14x^4 + 13x^3 - 11x^2 - 10x + 14$

$f(1) = \cancel{-14} - 14 + 13 - 11 - 10 \cancel{+14} = -22$

9. Find all zeros of $f(x) = 3x^3 - 4x^2 - 9x + 10$ given that one zero is 1.

$$\begin{array}{r|rrrr} 1 & 3 & -4 & -9 & 10 \\ & & 3 & -1 & -10 \\ \hline & 3 & -1 & -10 & 0 \end{array}$$

$\{-\frac{5}{3}, 1, 2\}$

$(3x^2 - x - 10)(x - 1) = 0$

$(3x + 5)(x - 2) = 0$

$x = -\frac{5}{3}$ $\quad x = 2$

$x = 1$

11. Show that $(x + 1)$ is a factor of $x^3 - 2x^2 + 3 = 0$. Use this information to find the solution set of this equation.

$$\begin{array}{r|rrrr} -1 & 1 & -2 & 0 & 3 \\ & & -1 & 3 & -3 \\ \hline & 1 & -3 & 3 & 0 \end{array}$$

$(x^2 - 3x + 3)(x + 1) = 0$

$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(3)}}{2(1)}$

$x = \frac{3 \pm \sqrt{-3}}{2} = \frac{3 \pm i\sqrt{3}}{2}$

$x = -1$

16. Given that $i$ is a multiple root of $x^4 + 2x^2 + 1 = 0$. Find the complete solution set.

$(x^2 + 1)^2 = 0$

$x^2 + 1 = 0$ $\quad$ $x^2 + 1 = 0$

$x^2 = -1$

$x = \pm i$ $\quad$ $x = \pm i$

{ i (multiplicity of 2), −i (multiplicity of 2) }

25. Find the roots of $f(x) = x^4 - 3x^2 - 28$.

$0 = (x^2 - 7)(x^2 + 4)$

$x^2 - 7 = 0$ $\quad$ $x^2 + 4 = 0$

$x = \pm\sqrt{7}$ $\quad$ $x = \pm 2i$

26. One zero of $f(x) = x^3 - x^2 - 5x + 21$ is $2 + i\sqrt{3}$. Find the remaining zeros.

Since this poly has real coefficients, $2 - i\sqrt{3}$ is also a zero.

$x^2$ – sum of roots $x$ + product of roots

$$\text{sum} = (2+i\sqrt{3}) + (2-i\sqrt{3}) = 4$$

$$\text{product} = (2+i\sqrt{3})(2-i\sqrt{3}) = 4 - i^2(3) = 4+3 = 7$$

$$x^2 - 4x + 7$$

$$\begin{array}{r|rrrr}
 & 1 & -1 & -5 & 21 \\
4 & & 4 & 12 & \\
-7 & & & -7 & -21 \\
\hline
 & 1 & 3 & 0 & 0
\end{array}$$

$$x + 3 = 0$$

$$x = -3$$

remaining zeros: $-3$, $2 - i\sqrt{3}$

# THE FACTOR THEOREM

L. Lewis

**THEOREM:** $f(x)$ is a polynomial function of degree $\geq 1$ and $x - c$ is a factor of $f$ if and only if $f(c) = 0$ (i.e. $c$ is a zero of $f$).

**EXAMPLE:** If $f(x) = 2x - 6$ and 3 is a zero of $f(x)$, then ______ is a factor of $f(x)$.

**EXAMPLE:** If $x + 1$ is a factor of $f(x)$, then _____ is a zero of $f(x)$.

**EXAMPLE:** If $f(x) = (x - 3)(x + 4)(2x - 1)$, then the zeros of $f(x)$ are: ____________________.

**EXAMPLE:** If $f(4) = 0$, then a factor of $f$ is: ______________.

**EXAMPLE:** If $3x - 1$ is a factor of $f(x)$, then $f(\quad) = 0$.

**Note: The complete factorization of a polynomial will include factors with ONLY integral coefficients.**

## PROBLEMS

1. Show 2 ways that $y - 1$ is a factor of $y^3 - 3y^2 + 3y - 1$.
2. Show 2 ways that $x - 2$ is a factor of $x^5 - 32$.
3. Factors of $x^3 + x^2 - 4x - 4$ are $(x - 2)$, $(x + 2)$ and $(x + 1)$. What are the zeros of the polynomial?
4. Given that the zeros of $x^3 - 6x^2 + 11x - 6$ are 1, 2, and 3. What are the factors of the polynomial? Check by multiplication.
5. If $f(x) = x^3 - x^2 + kx - 12$, find $k$ so that $f(x)$ is exactly divisible by $x + 3$.
6. If $f(x) = 4x^3 - 2x^2 - 3x + k$, find $k$ so that $f(x)$ is exactly divisible by $x - 2$.
7. If $f(x) = ax^5 + ax^4 + 13x^3 - 11x^2 - 10x - a$, find $f(1)$ if $f(-1) = 0$.
8. Show that $-3$ is a zero of $f(x) = x^3 + 7x^2 + 7x - 15$.
9. Find all zeros of $f(x) = 3x^3 - 4x^2 - 9x + 10$ given that one zero is 1.
10. One root of $x^3 + 8x^2 + 11x - 20 = 0$ is $-5$. Find the complete set of solutions of this equation.
11. Show that $(x + 1)$ is a factor of $x^3 - 2x^2 + 3 = 0$. Use this information to find the solution set of this equation.
12. One zero of $4x^3 - 11x^2 + 5x + 2$ is $-\frac{1}{4}$. Find the complete **factorization** of this polynomial and find the remaining two zeros.

## MULTIPLE ZEROS/ROOTS

13. a) Show that $3x + 1$ is a multiple factor of $p(x) = 27x^5 - 18x^3 - 8x^2 - x$.
    b) What is the multiplicity of the zero $-\frac{1}{3}$ ?
    c) Find all solutions of $p(x) = 0$.

14. a) Prove that 1 is a multiple root of $x^5 - 3x^4 + 8x^2 - 9x + 3 = 0$ and find its multiplicity.
    b) Use this information to solve the equation in part a).

15. The zeros of a polynomial $g(x)$ are $3$, $-2$, and $1$, where $1$ has a multiplicity of two.
    a) What is the degree of $g(x)$?
    b) What are the factors of $g(x)$ ?
    c) Find $g(x)$ in expanded form.

16. Given that $i$ is a multiple root of $x^4 + 2x^2 + 1 = 0$. Find the complete solution set.

17. If a polynomial equation of degree 5 has a solution set of $\{1\}$, what is the multiplicity of this root?

18. It is given that 1 is a multiple root of $x^5 - 3x^4 - 6x^3 + 26x^2 - 27x + 9 = 0$. Find all roots.

19. Solve $8x^5 - 12x^4 + 38x^3 - 49x^2 + 24x - 4 = 0$ if $\frac{1}{2}$ is a multiple root.

## NUMBER OF ROOTS - CONJUGATE PAIRS

20. One zero of a quadratic polynomial function with real coefficients is $3 + 4i$. What is the other zero?

21. If a cubic polynomial function with real coefficients has exactly one real zero, how many non-real zeros does it have?

22. Can a polynomial equation with real coefficients of odd degree have all imaginary roots? Explain.

23. Find the complete solution set of $x^4 - 2x^3 + 3x^2 - 8x - 4 = 0$ if one root is $2i$.

24. Find the complete solution set of $x^4 - 4x^3 + 6x^2 - 4x - 15 = 0$ if one root is $1 - 2i$.

25. Find the roots of $f(x) = x^4 - 3x^2 - 28$.

26. One zero of $f(x) = x^3 - x^2 - 5x + 21$ is $2 + i\sqrt{3}$. Find the remaining zeros.

# Turkey Trot Key

Sherwood

The sum of two positive numbers is 60. Find a function that models their product in terms of $x$, one of the numbers.

$0 < x < 60$

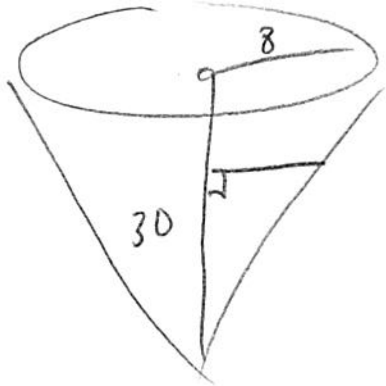
$x + y = 60$

$y = 60 - x$

$P(x) = x(60 - x)$

$P(x) = 60x - x^2$

ANS: $A(b) = \frac{b\sqrt{36-6b}}{2}$




$\frac{x}{h} = \frac{8}{30}$

$30x = 8h$

$\frac{30x}{8} = h$

$x > 0$

$x < 8$

Sellers

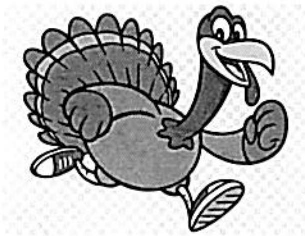
A water tank is in the shape of an inverted right circular cone with altitude 30 feet and radius 8 feet. The tank is filled to a depth of *h* feet. Let *x* be the radius of the circle at the top of the water level. Write a formula for the volume of the water as a function of *x*.

ANS: $P(x) = 60x - x^2$

$V = \frac{1}{3}\pi r^2 h$

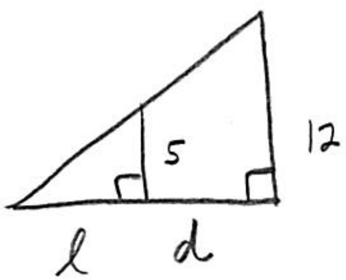
$V(x) = \frac{1}{\cancel{3}}\pi x^2\left(\frac{\overset{10}{\cancel{30}}x}{8}\right) = \frac{5}{4}\pi x^3$

$0 < x < 8$



Simon

A woman 5 ft tall is standing near a street lamp that is 12 ft tall. Find a functions that models the length of her shadow, *l,* in terms of her distance, *d,* from the base of the lamp.




$\ell(d) = \frac{5}{7}d$

$d > 0$

$\ell > 0$

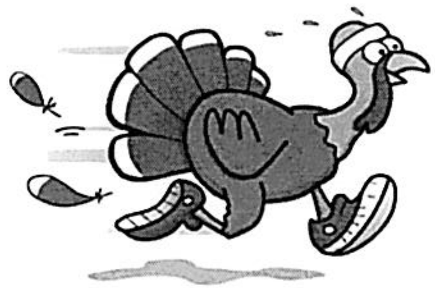
$\frac{\ell}{5} = \frac{\ell + d}{12}$

$12\ell = 5\ell + 5d$

$7\ell = 5d$

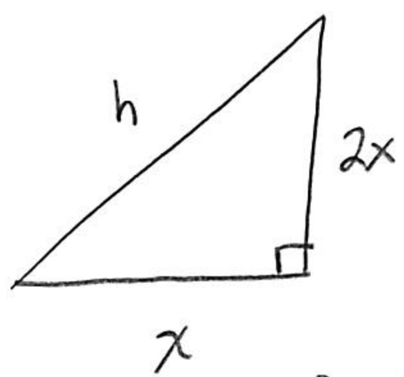
$\ell = \frac{5}{7}d$

ANS: $V(x) = \frac{5}{4}\pi x^3$



Carman

A right triangle has one leg twice as long as the other. Find a function that models its perimeter in terms of the length $x$ of the shorter leg.




$P(x) = x + 2x + h$

$P(x) = x + 2x + \sqrt{5x^2}$

$P(x) = 3x + \sqrt{5x^2} \quad , x > 0$

$= 3x + x\sqrt{5}$

ANS: $l(d) = \frac{5}{7}d$

$h^2 = (2x)^2 + x^2$

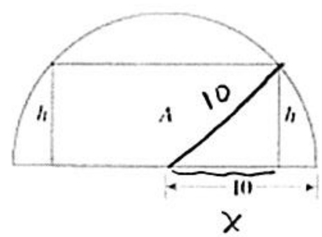
$h^2 = 5x^2$

$h = \pm\sqrt{5x^2}$

Loughran

A rectangle is inscribed in a semicircle of radius 10, as shown in the figure. Find a function that models the area of the rectangle in terms of its height *h*.

$0 < h < 10$




$A = lw$

$A(h) = l \cdot h$

$A(h) = 2\sqrt{100-h^2} \cdot h$

$A(h) = 2h\sqrt{100-h^2}$

ANS: $P(x) = 3x + x\sqrt{5}$

$x^2 + h^2 = 10^2$

$x^2 + h^2 = 100$

$x^2 = 100 - h^2$

$x = \pm\sqrt{100-h^2}$

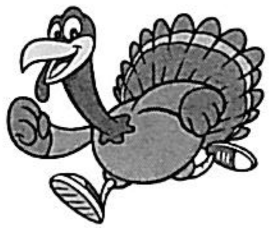
$h > 0$

$100 - h^2 > 0$

$(10-h)(10+h) > 0$

$-10 \qquad 10$

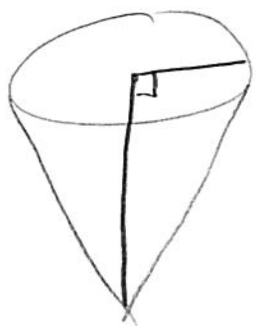
$-10 < h < 10$



Barnett

# The volume of a cone is 100 cubic inches. Find a function that models the height *h* of the cone in terms of its radius *r*.

$h(r) = \frac{300}{\pi r^2}$ $\quad r > 0$



ANS: $A(h) = 2h\sqrt{100 - h^2}$

$V = \frac{1}{3}\pi r^2 h$

$100 = \frac{1}{3}\pi r^2 h$

$300 = \pi r^2 h$

$h = \frac{300}{\pi r^2}$


Windwer

A wire 10 cm long is cut into two pieces, one of length $x$ and the other of length $10 - x$. Each is bent into the shape of a square. Find a function that models the total area enclosed by the two squares.

$x > 0$
$x < 10$

$\frac{x}{4}$ $\quad$ $\frac{10-x}{4}$

$A_{sq}(x) = \left(\frac{x}{4}\right)^2$ $\qquad$ $A(x) = \left(\frac{10-x}{4}\right)^2$

$$A(x) = \frac{x^2}{16} + \frac{(10-x)^2}{16}$$

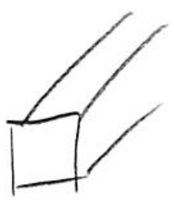
$$A(x) = \frac{x^2+100-20x+x^2}{16} = \frac{2x^2-20x+100}{16} = \frac{x^2}{8} - \frac{5x}{4} + \frac{25}{4}$$

$0 < x < 10$

ANS: $h(r) = \frac{300}{\pi r^2}$

Lee

An ***open*** box with a square base is to have a volume of 12 cubic feet. Find a function that models the surface area of the box.



$V = lwh$

$12 = lwh$

$12 = x \cdot x \cdot h$

$\frac{12}{x^2} = h$

$x > 0$

ANS: $A(x) = \frac{1}{8}x^2 - \frac{5}{4}x + \frac{25}{4}$

$S = x^2 + 2xh + 2xh$

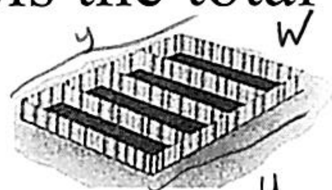
$S = x^2 + 4xh$

$S(x) = x^2 + 4\cancel{x}\left(\frac{12}{x^{\cancel{2}}}\right)$

$S(x) = x^2 + \frac{48}{x}$

Jacknis

A rancher with 750 ft of fencing wants to enclose a rectangular area then divide it into four pens with fencing parallel to one side of the rectangle (see figure). Find a function that models the total area of the four pens. in terms of w




of w

$w > 0$

$150 - w > 0$

$150 > w$

$5w + 2y = 750$

$5w = 750 - 2y$

$2y = 750 - 5w$

$y = \frac{750-5w}{2} = 375$

$A(x) = w\left(\frac{750-5w}{2}\right)$

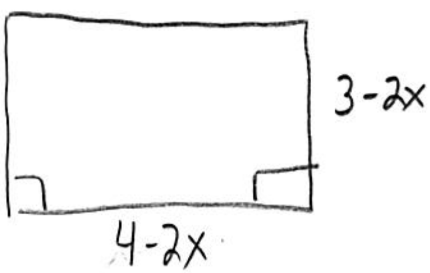
$A(x) = \frac{5w(150-w)}{2}$

$0 < w < 150$

ANS: $A(x) = \frac{48}{x} + x^2$

Stack

A sheet of cardboard 3 feet by 4 feet will be made into a box by cutting equal squares, of length $x$, from each corner and folding up the four edges. Express the volume of this box as a function of $x$.




ANS: $A(w) = \frac{5}{2}w(150 - w)$

$V(x) = x(3-2x)(4-2x)$

$V(x) = (3x-2x^2)(4-2x)$ $\quad 0 < x < 3/2$

$x > 0$

$4-2x > 0$
$4 > 2x$
$2 > x$

$3-2x > 0$
$3 > 2x$
$\frac{3}{2} > x$

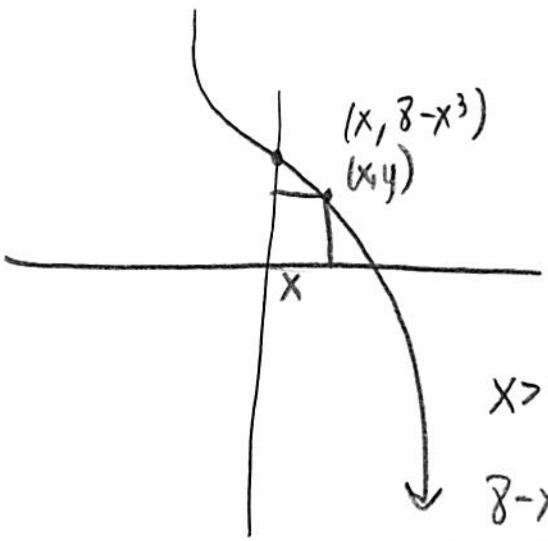
$V(x) = 12x - 8x^2 - 6x^2 + 4x^3$

$V(x) = 4x^3 - 14x^2 + 12x$

Ciavarella

# A rectangle is inscribed between the $x$-axis, the $y$-axis and the graph of $y=8-x^3$. Write the area of the rectangle as a function of $x$.

$y=-x^3+8$




$A(x)=x(8-x^3)$

$A(x)=8x-x^4$

$0<x<2$

$x>0$

$8-x^3>0$

$(2-x)$

ANS: $V(x)=4x^3-14x^2+12x$


Edelman

A manufacturer makes a metal can that holds 1L or 1000 cubic centimeters. Write a formula for the surface area of the can in terms of its radius.

$1000 = \pi r^2 h$

$\frac{1000}{\pi r^2} = h$

$r > 0$

$S = 2\pi r^2 + 2\pi r h$

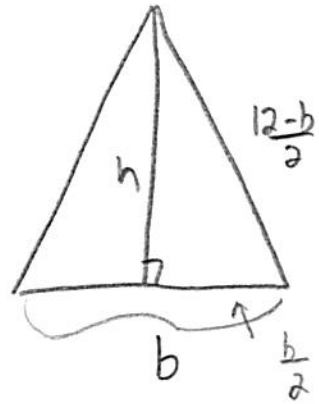
$S(r) = 2\pi r^2 + 2\cancel{\pi r}\left(\frac{1000}{\cancel{\pi} r^{\cancel{2}}}\right)$

$r > 0$

ANS: $A(x) = -x^4 + 8x$

$S(r) = 2\pi r^2 + \frac{2000}{r}$

$\frac{12-b}{2}$




$$h^2 + \left(\frac{b}{2}\right)^2 = \left(\frac{12-b}{2}\right)^2$$

$$h^2 + \frac{b^2}{4} = \frac{b^2 - 24b + 144}{4}$$

$$h^2 = \frac{\cancel{b^2} - 24b + 144 \cancel{- b^2}}{4}$$

$$h^2 = \frac{144 - 24b}{4}$$

$$h = \pm\sqrt{36-6b}$$

Callahan

$$h^2 = 36 - 6b$$

# An isosceles triangle has a perimeter of 12 cm. Find a function that models its area in terms of the length of its base *b*.

$$A = \frac{1}{2}bh \qquad 0 < b < 6$$

$$36 - 6b > 0$$
$$36 > 6b$$
$$6 > b$$

$$A(b) = \frac{1}{2}b\left(\sqrt{36-6b}\right)$$

ANS: $S(r) = 2\pi r^2 + \frac{2000}{r}$