Name: ______________________________ Date: __________

PCH: Rational Zeros and Intermediate Value Theorems Ms.Loughran

Do Now:

1. When a function $f(x)$ is divided by $2x-3$, the quotient is $3x^2-4x+2$ and remainder is -7. Find $f(x)$ in simplest form.

divisor · quotient + remainder

$f(x) = (2x-3)(3x^2-4x+2) - 7$

$6x^3 - 8x^2 + 4x - 9x^2 + 12x - 6 - 7$

$6x^3 - 17x^2 + 16x - 6 - 7$

$6x^3 - 17x^2 + 16x - 13$

2. Find the remainder when $x^{124} - 5x^{76} + 2x^{45} - 3x + 5$ is divided by $x+1$.

$(-1)^{124} - 5(-1)^{76} + 2(-1)^{45} - 3(-1) + 5$

$1 \cancel{-5} - 2 + 3 \cancel{+5} = 2$

Rational Zeros Theorem

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ has integer coefficients, then every rational zero of P is of the form

$$\frac{p}{q}$$

where p is a factor of the constant coefficient a_0

and q is a factor of the leading coefficient a_n

$$\begin{array}{r|rrrrr} 2 & 1 & -5 & -5 & 23 & 10 \\ & & 2 & -6 & -22 & 2 \\ \hline & 1 & -3 & -11 & 1 & 12 \end{array} \qquad \begin{array}{r|rrrrr} -2 & 1 & -5 & -5 & 23 & 10 \\ & & -2 & 14 & -18 & -10 \\ \hline & 1 & -7 & 9 & 5 & 0 \end{array} \qquad \begin{array}{r|rrrrr} 5 & 1 & -5 & -5 & 23 & 10 \\ & & 5 & 0 & -25 & -10 \\ \hline & 1 & 0 & -5 & -2 & 0 \end{array}$$

Classwork:

1. Let $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$. Find the zeros of $P(x)$.

possible rational zeros: $\text{prz} = \dfrac{\pm 1, \pm 2, \pm 5, \pm 10}{\pm 1} = \pm 1, \pm 2, \pm 5, \pm 10$

$P(1) = 1 - \cancel{5} - \cancel{5} + 23 + 10 \neq 0$

$P(-1) = 1 + \cancel{5} - \cancel{5} - 23 + 10 \neq 0$

$$\begin{array}{r|rrrrr} -2 & 1 & -5 & -5 & 23 & 10 \\ & & -2 & 14 & -18 & -10 \\ \hline 5 & 1 & -7 & 9 & 5 & \cancel{0} \\ & & 5 & -10 & -5 & \\ \hline & 1 & -2 & -1 & 0 & \end{array}$$

$(x+2)(x-5)(x^2-2x-1) = 0$

$x = -2 \quad \Big| \quad x = 5 \quad \Big| \quad x^2 - 2x - 1 = 0$

$x^2 - 2x + 1 = 1 + 1$

$(x-1)^2 = 2$

$x - 1 = \pm\sqrt{2}$

$x = 1 \pm \sqrt{2}$

$\{-2, 5, 1 \pm \sqrt{2}\}$

2. Factor the polynomial $P(x) = 2x^3 + x^2 - 13x + 6$

$\text{prz}: \dfrac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

$P(1) = 2 + 1 - 13 + 6 \neq 0$

$P(-1) = -2 + 1 + 13 + 6 \neq 0$

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$(x-2)(2x^2 + 5x - 3)$

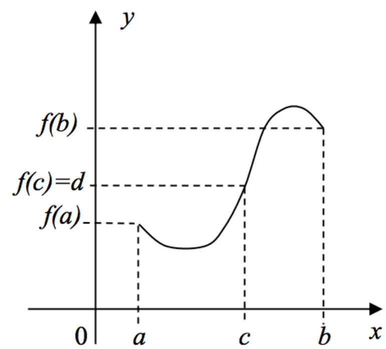
$(x-2)(2x-1)(x+3)$

For 3 - 8, find the complete factorization and all zeros of the following polynomials using the information given.

3. $P(x) = 2x^5 - 5x^4 + x^3 + 4x^2 - 4x$

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.



This theorem helps locate the real zeros of a polynomial function. If $f(a)$ is positive real number, and another $f(b)$ is a negative number and $a < b$, you can conclude that the function has at least one real zero between these two variables

IVT

9. Use the Intermediate Value Theorem to prove that a zero exists on the interval $[1,2]$ of the function $f(x) = -x^3 + 2x^2 + 9x - 11$.

$f(1) = -1 + 2 + 9 - 11 = -1$

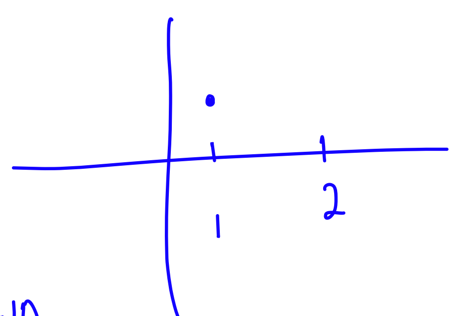
$f(2) = -8 + 8 + 18 - 11 = 7$

By the IVT if $f(1) < 0$ and $f(2) > 0$ then $f(x) = 0$ at some pt in the interval

10. Use the Intermediate Value Theorem to prove that $f(x) = x^3 + x$ takes on the value 9 for some x in $[1,2]$.

$f(1) = 2$

$f(2) = 8 + 2 = 10$



By the IVT since $f(1) = 2$ and $f(2) = 10$ $f(x) = 9$ at some pt in that interval.

11. Selected value of the continuous function f are shown in the table below. Is the following statement true or false?

$f(x) = 2$ has at least 1 solution in the interval $[0,7]$.

True

x	$f(x)$
0	4
3	1
4	−4
5	−12
7	−32

Since $f(0) = 4$ and $f(3) = 1$ there has to be a place in the interval $[0, 3]$ where $f(x) = 2$ by the IVT

12. Selected value of the continuous function f are shown in the table below. Is the following statement true or false?

$f(x) = 5$ has at least 1 solution in the interval $[-3,2]$.

True

x	$f(x)$
−3	−2
0	10
1	11
2	8

If $f(-3) = -2$ and $f(0) = 10$ then by the IVT $f(x) = 5$ in the interval $[-3, 0]$

Homework 11-27

THE FACTOR THEOREM

L. Lewis

THEOREM: $f(x)$ is a polynomial function of degree ≥ 1 and $x - c$ is a factor of f if and only if $f(c) = 0$ (i.e. c is a zero of f).

EXAMPLE: If $f(x) = 2x - 6$ and 3 is a zero of $f(x)$, then ______ is a factor of $f(x)$.

EXAMPLE: If $x + 1$ is a factor of $f(x)$, then ______ is a zero of $f(x)$.

EXAMPLE: If $f(x) = (x - 3)(x + 4)(2x - 1)$, then the zeros of $f(x)$ are: ____________________.

EXAMPLE: If $f(4) = 0$, then a factor of f is:____________.

EXAMPLE: If $3x - 1$ is a factor of $f(x)$, then $f(\quad) = 0$.

Note: The complete factorization of a polynomial will include factors with ONLY integral coefficients.

PROBLEMS

(2) $2^5 - 32 = 0$

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & 0 \end{array}$$

1. Show 2 ways that $y - 1$ is a factor of $y^3 - 3y^2 + 3y - 1$.
2. Show 2 ways that $x - 2$ is a factor of $x^5 - 32$.
3. Factors of $x^3 + x^2 - 4x - 4$ are $(x - 2)$, $(x + 2)$ and $(x + 1)$. What are the zeros of the polynomial?
4. Given that the zeros of $x^3 - 6x^2 + 11x - 6$ are 1, 2, and 3. What are the factors of the polynomial? Check by multiplication.
   $(x-1)(x-2)(x-3)$
5. If $f(x) = x^3 - x^2 + kx - 12$, find k so that $f(x)$ is exactly divisible by $x + 3$.
6. If $f(x) = 4x^3 - 2x^2 - 3x + k$, find k so that $f(x)$ is exactly divisible by $x - 2$. $k = -18$
7. If $f(x) = ax^5 + ax^4 + 13x^3 - 11x^2 - 10x - a$, find $f(1)$ if $f(-1) = 0$.
8. Show that -3 is a zero of $f(x) = x^3 + 7x^2 + 7x - 15$.
   $f(-3) = (-3)^3 + 7(-3)^2 + 7(-3) - 15 = -27 + 63 - 21 - 15 = 0$
   or you could do syn div. + show remainder is 0
9. Find all zeros of $f(x) = 3x^3 - 4x^2 - 9x + 10$ given that one zero is 1.
10. One root of $x^3 + 8x^2 + 11x - 20 = 0$ is -5. Find the complete set of solutions of this equation.
    $\{-4, 1, -5\}$
11. Show that $(x + 1)$ is a factor of $x^3 - 2x^2 + 3 = 0$. Use this information to find the solution set of this equation.
12. One zero of $4x^3 - 11x^2 + 5x + 2$ is $-¼$. Find the complete factorization of this polynomial and find the remaining two zeros.
    $(4x+1)(x-1)(x-2)$
    remaining zeros: 1, 2

(14)

$$\begin{array}{r|rrrrrr} 1 & 1 & -3 & 0 & 8 & -9 & 3 \\ & & 1 & -2 & -2 & 6 & -3 \\ \hline 1 & 1 & -2 & -2 & 6 & -3 & 0 \\ & & 1 & -1 & -3 & 3 & \end{array}$$

$(x-1)^3(x^2-3) = 0$

$x^2 - 3 = 0$

$$\begin{array}{r|rrrrr} 1 & 1 & -1 & -3 & 3 & 0 \\ & & 1 & 0 & -3 & \\ \hline & 1 & 0 & -3 & 0 & \end{array}$$

$x = 1$ (triple)

$x^2 = 3$

$x = \pm\sqrt{3}$

MULTIPLE ZEROS/ROOTS

13. a) Show that $3x + 1$ is a multiple factor of $p(x) = 27x^5 - 18x^3 - 8x^2 - x$.
    b) What is the multiplicity of the zero $-\frac{1}{3}$?
    c) Find all solutions of $p(x) = 0$.

14. a) Prove that 1 is a multiple root of $x^5 - 3x^4 + 8x^2 - 9x + 3 = 0$ and find its multiplicity.
    b) Use this information to solve the equation in part a). $\{1$ (multiplicity of 3), $\pm\sqrt{3}\}$

15. The zeros of a polynomial $g(x)$ are 3, -2, and 1, where 1 has a multiplicity of two.
    a) What is the degree of $g(x)$?
    b) What are the factors of $g(x)$?
    c) Find $g(x)$ in expanded form.

16. Given that i is a multiple root of $x^4 + 2x^2 + 1 = 0$. Find the complete solution set.

17. If a polynomial equation of degree 5 has a solution set of $\{1\}$, what is the multiplicity of this root?

18. It is given that 1 is a multiple root of $x^5 - 3x^4 - 6x^3 + 26x^2 - 27x + 9 = 0$. Find all roots.
    1 (multiplicity of 3), ± 3

19. Solve $8x^5 - 12x^4 + 38x^3 - 49x^2 + 24x - 4 = 0$ if $\frac{1}{2}$ is a multiple root.

NUMBER OF ROOTS - CONJUGATE PAIRS

20. One zero of a quadratic polynomial function with real coefficients is $3 + 4i$. What is the other zero?
    $3 - 4i$

21. If a cubic polynomial function with real coefficients has exactly one real zero, how many non-real zeros does it have?

22. Can a polynomial equation with real coefficients of odd degree have all imaginary roots? Explain. No
    If the coefficients of the polynomial are real, then imaginary roots come in conjugate pairs.

23. Find the complete solution set of $x^4 - 2x^3 + 3x^2 - 8x - 4 = 0$ if one root is $2i$.

24. Find the complete solution set of $x^4 - 4x^3 + 6x^2 - 4x - 15 = 0$ if one root is $1 - 2i$.
    $\{1 \pm 2i, 3, -1\}$

25. Find the roots of $f(x) = x^4 - 3x^2 - 28$.

26. One zero of $f(x) = x^3 - x^2 - 5x + 21$ is $2 + i\sqrt{3}$. Find the remaining zeros.

(18)

$$\begin{array}{r|rrrrrr} 1 & 1 & -3 & -6 & 26 & -27 & 9 \\ & & 1 & -2 & -8 & 18 & -9 \\ \hline 1 & 1 & -2 & -8 & 18 & -9 & 0 \\ & & 1 & -1 & -9 & 9 & \\ \hline 1 & 1 & -1 & -9 & 9 & 0 & \\ & & 1 & 0 & -9 & & \\ \hline \end{array}$$

$1 \quad 0 \quad -9 \quad 0$

$(x-1)^3(x^2-9)=0$

$x=1, \quad x=\pm 3$

(24) $1+2i$

sum $=(1-2i)+(1+2i)=2$

product $=(1-2i)(1+2i) = 1-4i^2 = 5$

$x^2 - \text{sum}\, x + \text{product}$

x^2-2x+5

$$
\begin{array}{r|rrrrr}
 & 1 & -4 & 6 & -4 & -15 \\
2 & & 2 & -4 & -6 & \\
-5 & & & -5 & 10 & 15 \\
\hline
 & 1 & -2 & -3 & 0 & 0
\end{array}
$$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3, x=-1, \quad x=1\pm 2i$