Name:____________________________________ Date:_________

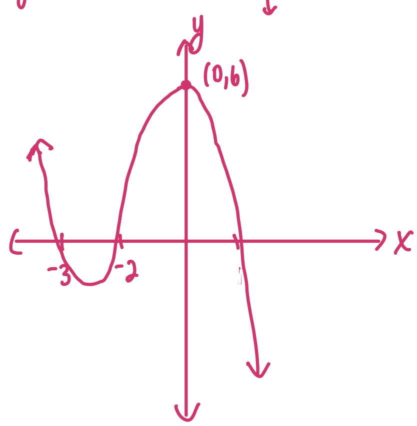
PCH: Sketching Polynomials without a Graphing Calculator Ms. Loughran

Do Now:

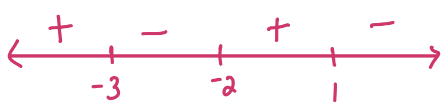
Sketch the general graph of each function without your graphing calculator. Your sketch should contain both the *x*- and *y*-intercepts and indicate the end behavior of the graph.

1. $f(x) = -(x+3)(x+2)(x-1)$

degree: 3 odd



f(x)



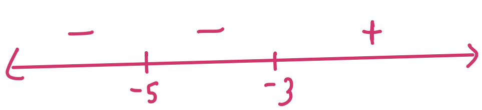
x-intercepts: (-3,0), (-2,0), (1,0)

y-intercept: (0,6)

2. $f(x) = (x+5)^2(x+3)$

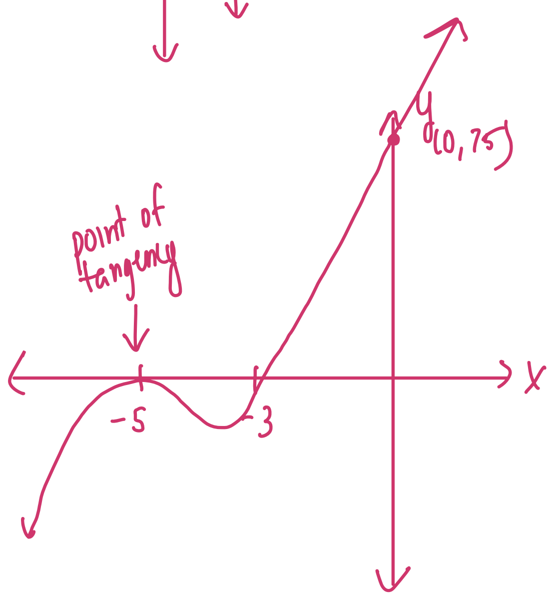
degree: 3 odd

f(x)



x-int: (-5,0), (-3,0)

y-int: (0,75)



3. $f(x) = x^3 + 2x^2 - 36x - 72$

$f(x) = x^2(x+2) - 36(x+2)$

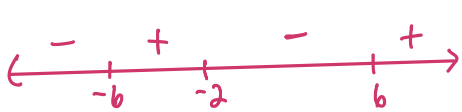
$f(x) = (x^2 - 36)(x+2)$

$f(x) = (x+6)(x-6)(x+2)$

degree: 3 (odd)

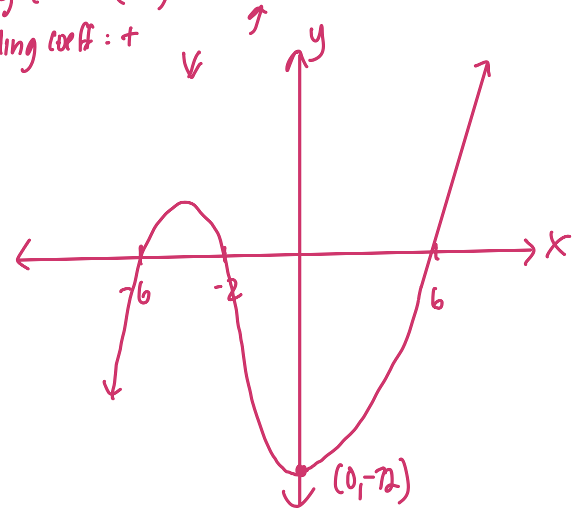
leading coeff: +

f(x)



x-int: (±6,0), (-2,0)

y-int: (0,-72)



POLYNOMIAL GRAPH SUMMARY

The following are general statements about the graph of a polynomial function $y = P(x)$.

1. A polynomial function is continuous — i.e. it is defined for all real values of x and can be drawn without lifting the pencil from the paper. Therefore, it has no holes or vertical asymptotes, nor does it have any "jumps" or other gaps.

2. The "turns" taken by the graph are smooth. There are no sharp corners or points.

3. As $|x|$ gets very large, the points of the graph move further and further away from the x-axis. And, there are no horizontal or oblique asymptotes.

The following are statements about the roots of the polynomial equation $P(x) = 0$ with real coefficients and the curve $y = P(x)$.

1. A polynomial equation of degree n has n roots, real and/or nonreal, counting multiplicity.

2. An intersection (not tangency or terrace point) of the curve and the x-axis indicates one real root.

3. A point of tangency located on the x-axis indicates a real root of even multiplicity (2 equal roots, 4 equal roots, etc.).

4. A terrace point located on the x-axis indicates a real root of odd multiplicity (3 equal roots, 5 equal roots, etc.).

5. Non-real roots occur in conjugate pairs.

6. A relative maximum below the x-axis, or a relative minimum above the x-axis, or a terrace point above or below the x-axis indicates one pair of conjugate nonreal roots or a multiplicity of conjugate nonreal roots.
NOTE: The converse of this statement is false (consider $y = x^4 - 1$).

Graphs of Polynomial Functions: End behavior

think of x^3

think of x^2

	Odd degree		Even degree	
Sign of Leading Coefficient	Positive	Negative	Positive	Negative
End behavior				

0 is a triple root

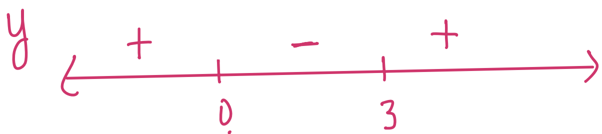
From the Sketching Polynomials section of today's packet

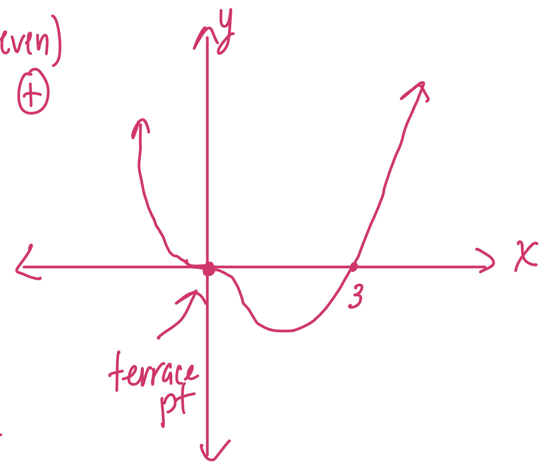
9. $y = x^3(x - 3)$

degree: 4 (even)

leading coefficient $\oplus$

↑ ↑





x-int: (0,0), (3,0)

y-int: (0,0)

triple zero
terrace point

* you have a terrace point when you have a zero with odd multiplicity greater than one *

Homework 11-28

p. 271

67 a) $P(-1) = 6(-1)^{1000} - 17(-1)^{562} + 12(-1) + 26 = 3$

b) $P(1) = 1^{567} - 3(1)^{400} + 1^{9} + 2 \neq 0$

so $x-1$ is <u>not</u> a factor of $x^{567} - 3x^{400} + x^{9} + 2$

pp. 298-299

4 a) $\left\{ 0, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right\}$

b) $x(x^2 + x + 1)$

6 a) $\{ \pm i, \pm\sqrt{2} \}$

b) $(x^2+1)(x^2-2)$

8 a) $\{ i\sqrt{3}$ (multiplicity of 2), $-i\sqrt{3}$ (mult. of 2) $\}$

b) $(x^2+3)^2$

12 a) $\left\{ -1, 2, -1 \pm i\sqrt{3}, \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right\}$

b) $(x+1)(x-2)(x^2+2x+4)(x^2-x+1)$

$x^6 - 7x^3 - 8$

$(x^3-8)(x^3+1)$

then use cubes formulas

tweak AM method

37 $R(x) = (x-1)^2(x^2-2x+1)$

$R(x) = x^4 - 4x^3 + 10x^2 - 12x + 5$

44 $\{ -3, -2 \pm i\sqrt{2} \}$

47 $\left\{ -\frac{3}{2}, -1 \pm i\sqrt{2} \right\}$

49 $\{ 1, -2, \pm 3i \}$

31. $\dfrac{x^5 + 3x^3 - 6}{x - 1}$

32. $\dfrac{x^3 - 9x^2 + 27x - 27}{x - 3}$

33. $\dfrac{2x^3 + 3x^2 - 2x + 1}{x - \frac{1}{2}}$

34. $\dfrac{6x^4 + 10x^3 + 5x^2 + x + 1}{x + \frac{2}{3}}$

35. $\dfrac{x^3 - 27}{x - 3}$

36. $\dfrac{x^4 - 16}{x + 2}$

37–49 ■ Use synthetic division and the Remainder Theorem to evaluate $P(c)$.

37. $P(x) = 4x^2 + 12x + 5, \quad c = -1$

38. $P(x) = 2x^2 + 9x + 1, \quad c = \frac{1}{2}$

39. $P(x) = x^3 + 3x^2 - 7x + 6, \quad c = 2$

40. $P(x) = x^3 - x^2 + x + 5, \quad c = -1$

41. $P(x) = x^3 + 2x^2 - 7, \quad c = -2$

42. $P(x) = 2x^3 - 21x^2 + 9x - 200, \quad c = 11$

43. $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14, \quad c = -7$

44. $P(x) = 6x^5 + 10x^3 + x + 1, \quad c = -2$

45. $P(x) = x^7 - 3x^2 - 1, \quad c = 3$

46. $P(x) = -2x^6 + 7x^5 + 40x^4 - 7x^2 + 10x + 112, \quad c = -3$

47. $P(x) = 3x^3 + 4x^2 - 2x + 1, \quad c = \frac{2}{3}$

48. $P(x) = x^3 - x + 1, \quad c = \frac{1}{4}$

49. $P(x) = x^3 + 2x^2 - 3x - 8, \quad c = 0.1$

50. Let

$$P(x) = 6x^7 - 40x^6 + 16x^5 - 200x^4 - 60x^3 - 69x^2 + 13x - 139$$

Calculate $P(7)$ by **(a)** using synthetic division and **(b)** substituting $x = 7$ into the polynomial and evaluating directly.

51–54 ■ Use the Factor Theorem to show that $x - c$ is a factor of $P(x)$ for the given value(s) of c.

51. $P(x) = x^3 - 3x^2 + 3x - 1, \quad c = 1$

52. $P(x) = x^3 + 2x^2 - 3x - 10, \quad c = 2$

53. $P(x) = 2x^3 + 7x^2 + 6x - 5, \quad c = \frac{1}{2}$

54. $P(x) = x^4 + 3x^3 - 16x^2 - 27x + 63, \quad c = 3, -3$

55–56 ■ Show that the given value(s) of c are zeros of $P(x)$, and find all other zeros of $P(x)$.

55. $P(x) = x^3 - x^2 - 11x + 15, \quad c = 3$

56. $P(x) = 3x^4 - x^3 - 21x^2 - 11x + 6, \quad c = \frac{1}{3}, -2$

57–60 ■ Find a polynomial of the specified degree that has the given zeros.

57. Degree 3; zeros $-1, 1, 3$

58. Degree 4; zeros $-2, 0, 2, 4$

59. Degree 4; zeros $-1, 1, 3, 5$

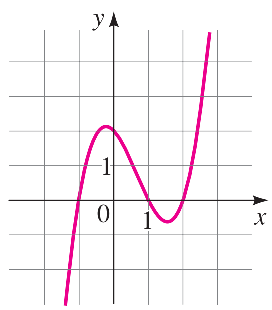
60. Degree 5; zeros $-2, -1, 0, 1, 2$

61. Find a polynomial of degree 3 that has zeros 1, -2, and 3, and in which the coefficient of x^2 is 3.

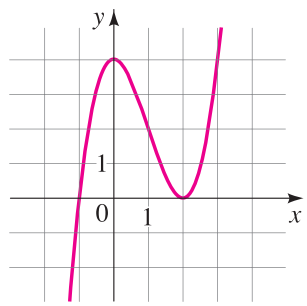
62. Find a polynomial of degree 4 that has integer coefficients and zeros 1, -1, 2, and $\frac{1}{2}$.

63–66 ■ Find the polynomial of the specified degree whose graph is shown.

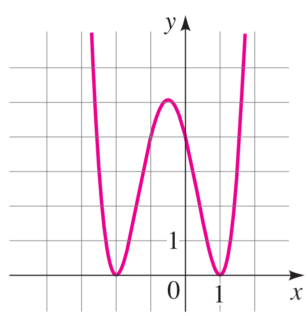
63. Degree 3



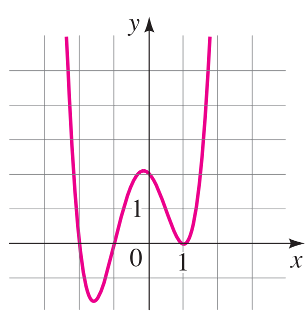
64. Degree 3



65. Degree 4



66. Degree 4



Discovery • Discussion

67. Impossible Division? Suppose you were asked to solve the following two problems on a test:

A. Find the remainder when $6x^{1000} - 17x^{562} + 12x + 26$ is divided by $x + 1$.

B. Is $x - 1$ a factor of $x^{567} - 3x^{400} + x^9 + 2$?

Obviously, it's impossible to solve these problems by dividing, because the polynomials are of such large degree. Use one or more of the theorems in this section to solve these problems *without* actually dividing.

Example 8 Factoring a Polynomial into Linear and Quadratic Factors

Let $P(x) = x^4 + 2x^2 - 8$.

(a) Factor P into linear and irreducible quadratic factors with real coefficients.

(b) Factor P completely into linear factors with complex coefficients.

Solution

(a)

$$\begin{aligned} P(x) &= x^4 + 2x^2 - 8 \\ &= (x^2 - 2)(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \end{aligned}$$

The factor $x^2 + 4$ is irreducible since it has only the imaginary zeros $\pm 2i$.

(b) To get the complete factorization, we factor the remaining quadratic factor.

$$\begin{aligned} P(x) &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x - 2i)(x + 2i) \end{aligned}$$

■

3.5 Exercises

1–12 ■ A polynomial P is given.

(a) Find all zeros of P, real and complex.

(b) Factor P completely.

1. $P(x) = x^4 + 4x^2$

2. $P(x) = x^5 + 9x^3$

3. $P(x) = x^3 - 2x^2 + 2x$

4. $P(x) = x^3 + x^2 + x$

5. $P(x) = x^4 + 2x^2 + 1$

6. $P(x) = x^4 - x^2 - 2$

7. $P(x) = x^4 - 16$

8. $P(x) = x^4 + 6x^2 + 9$

9. $P(x) = x^3 + 8$

10. $P(x) = x^3 - 8$

11. $P(x) = x^6 - 1$

12. $P(x) = x^6 - 7x^3 - 8$

13–30 ■ Factor the polynomial completely and find all its zeros. State the multiplicity of each zero.

13. $P(x) = x^2 + 25$

14. $P(x) = 4x^2 + 9$

15. $Q(x) = x^2 + 2x + 2$

16. $Q(x) = x^2 - 8x + 17$

17. $P(x) = x^3 + 4x$

18. $P(x) = x^3 - x^2 + x$

19. $Q(x) = x^4 - 1$

20. $Q(x) = x^4 - 625$

21. $P(x) = 16x^4 - 81$

22. $P(x) = x^3 - 64$

23. $P(x) = x^3 + x^2 + 9x + 9$

24. $P(x) = x^6 - 729$

25. $Q(x) = x^4 + 2x^2 + 1$

26. $Q(x) = x^4 + 10x^2 + 25$

27. $P(x) = x^4 + 3x^2 - 4$

28. $P(x) = x^5 + 7x^3$

29. $P(x) = x^5 + 6x^3 + 9x$

30. $P(x) = x^6 + 16x^3 + 64$

31–40 ■ Find a polynomial with integer coefficients that satisfies the given conditions.

31. P has degree 2, and zeros $1 + i$ and $1 - i$.

32. P has degree 2, and zeros $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$.

33. Q has degree 3, and zeros 3, $2i$, and $-2i$.

34. Q has degree 3, and zeros 0 and i.

35. P has degree 3, and zeros 2 and i.

36. Q has degree 3, and zeros -3 and $1 + i$.

37. R has degree 4, and zeros $1 - 2i$ and 1, with 1 a zero of multiplicity 2.

38. S has degree 4, and zeros $2i$ and $3i$.

39. T has degree 4, zeros i and $1 + i$, and constant term 12.

40. U has degree 5, zeros $\frac{1}{2}$, -1, and $-i$, and leading coefficient 4; the zero -1 has multiplicity 2.

41–58 ■ Find all zeros of the polynomial.

41. $P(x) = x^3 + 2x^2 + 4x + 8$

42. $P(x) = x^3 - 7x^2 + 17x - 15$

43. $P(x) = x^3 - 2x^2 + 2x - 1$

44. $P(x) = x^3 + 7x^2 + 18x + 18$

45. $P(x) = x^3 - 3x^2 + 3x - 2$

46. $P(x) = x^3 - x - 6$

47. $P(x) = 2x^3 + 7x^2 + 12x + 9$

48. $P(x) = 2x^3 - 8x^2 + 9x - 9$

49. $P(x) = x^4 + x^3 + 7x^2 + 9x - 18$

50. $P(x) = x^4 - 2x^3 - 2x^2 - 2x - 3$

51. $P(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$

52. $P(x) = x^5 + x^3 + 8x^2 + 8$ [*Hint:* Factor by grouping.]

53. $P(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$

54. $P(x) = x^4 - x^2 + 2x + 2$

55. $P(x) = 4x^4 + 4x^3 + 5x^2 + 4x + 1$

56. $P(x) = 4x^4 + 2x^3 - 2x^2 - 3x - 1$

57. $P(x) = x^5 - 3x^4 + 12x^3 - 28x^2 + 27x - 9$

58. $P(x) = x^5 - 2x^4 + 2x^3 - 4x^2 + x - 2$

59–64 ■ A polynomial P is given.

(a) Factor P into linear and irreducible quadratic factors with real coefficients.

(b) Factor P completely into linear factors with complex coefficients.

59. $P(x) = x^3 - 5x^2 + 4x - 20$

60. $P(x) = x^3 - 2x - 4$

61. $P(x) = x^4 + 8x^2 - 9$

62. $P(x) = x^4 + 8x^2 + 16$

63. $P(x) = x^6 - 64$

64. $P(x) = x^5 - 16x$

65. By the Zeros Theorem, every nth-degree polynomial equation has exactly n solutions (including possibly some that are repeated). Some of these may be real and some may be imaginary. Use a graphing device to determine how many real and imaginary solutions each equation has.

(a) $x^4 - 2x^3 - 11x^2 + 12x = 0$

(b) $x^4 - 2x^3 - 11x^2 + 12x - 5 = 0$

(c) $x^4 - 2x^3 - 11x^2 + 12x + 40 = 0$

66–68 ■ So far we have worked only with polynomials that have real coefficients. These exercises involve polynomials with real and imaginary coefficients.

66. Find all solutions of the equation.

(a) $2x + 4i = 1$

(b) $x^2 - ix = 0$

(c) $x^2 + 2ix - 1 = 0$

(d) $ix^2 - 2x + i = 0$

67. **(a)** Show that $2i$ and $1 - i$ are both solutions of the equation

$$x^2 - (1 + i)x + (2 + 2i) = 0$$

but that their complex conjugates $-2i$ and $1 + i$ are not.

(b) Explain why the result of part (a) does not violate the Conjugate Zeros Theorem.

68. **(a)** Find the polynomial with *real* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and in which the coefficient of the highest power is 1.

(b) Find the polynomial with *complex* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and in which the coefficient of the highest power is 1.

Discovery • Discussion

69. Polynomials of Odd Degree The Conjugate Zeros Theorem says that the complex zeros of a polynomial with real coefficients occur in complex conjugate pairs. Explain how this fact proves that a polynomial with real coefficients and odd degree has at least one real zero.

70. Roots of Unity There are two square roots of 1, namely 1 and -1. These are the solutions of $x^2 = 1$. The fourth roots of 1 are the solutions of the equation $x^4 = 1$ or $x^4 - 1 = 0$. How many fourth roots of 1 are there? Find them. The cube roots of 1 are the solutions of the equation $x^3 = 1$ or $x^3 - 1 = 0$. How many cube roots of 1 are there? Find them. How would you find the sixth roots of 1? How many are there? Make a conjecture about the number of nth roots of 1.

3.6 Rational Functions

A rational function is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We assume that $P(x)$ and $Q(x)$ have no factor in common. Even though rational functions are constructed from polynomials, their graphs look quite different than the graphs of polynomial functions.

6. $P(x) = x^4 - x^2 - 2$

$P(x) = (x^2 - 2)(x^2 + 1)$

$0 = (x^2 - 2)(x^2 + 1)$

$x^2 = 2$ | $x^2 = -1$

$x = \pm\sqrt{2}$ | $x = \pm i$

44 prz: $\frac{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18}{\pm 1} = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

$$\begin{array}{r|rrrr} -3 & 1 & 7 & 18 & 18 \\ & & -3 & -12 & -18 \\ \hline & 1 & 4 & 6 & 0 \end{array}$$

$(x^2 + 4x + 6)(x + 3) = 0$

$x^2 + 4x + 6 = 0$ | $x = -3$

$x^2 + 4x + 4 = -6 + 4$

$(x+2)^2 = -2$

$x + 2 = \pm\sqrt{-2}$

$x = -2 \pm i\sqrt{2}$

$1 \pm 2i$

sum = 2

product = $1 - 4i^2 = 5$

37. R has degree 4, and zeros $1 - 2i$ and 1, with 1 a zero of multiplicity 2.

$R(x) = (x-1)^2(x^2 - 2x + 5)$