2019

[Remember the parts in brackets are notes for you & do not have to be written in your solutions.]

1) (a)

$$\int_0^5 E(t)\,dt = 153.457\ldots$$

153 ← [directions said round to nearest whole #]

153 fish enter the lake from midnight to 5 am

(b)

$$\frac{\int_0^5 L(t)\,dt}{5-0} = 6.05903\ldots$$

[used average value of function formula]

The avg # of fish leaving the lake per hr from midnight to 5 AM is 6.059 fish per hour.

(c) [Use Candidate Test → candidates are endpts of interval and at critical pts of the function]

Let $N(t)$ = # of fish in lake at time t

$$N(t) = \int (E(t) - L(t))\,dt$$

$$N'(t) = E(t) - L(t)$$

look at graph of $N'(t)$ to find info about $N(t)$
from $0 \le t \le 8$

Since $N'(t)$ is positive from $0 < t < 6.204$ $N(t)$ is increasing so we know $N(6.204)$ is greater than $N(0)$.
Since $N'(t)$ is negative from $6.204 < t < 8$ $N(t)$ is decreasing so we know $N(6.204)$ is greater than $N(8)$

This means $N(t)$ has its max value when $t = 6.204$ hours.

(d) $N'(t)$ = rate of change of # of fish in lake

$N(t) = \int (E(t) - L(t))\,dt$

$N'(t) = E(t) - L(t)$

$N''(t) = E'(t) - L'(t)$

$N''(5) = E'(5) - L'(5) = -10.7228$

Since $N''(5) < 0$, $N'(t)$ is decreasing at $t = 5$.

(2)

(a) Since v_p is differentiable & therefore continuous, then by MVT, there is guaranteed a value in the interval $0.3 < t < 2.8$ at which

$$v_p'(t) = \frac{v_p(2.8) - v_p(0.3)}{2.8 - 0.3} = \frac{55 - 55}{2.5} = 0$$

(b) $\int_0^{2.8} v_p(t)\,dt \approx$

$\frac{1}{2}(0.3(0+55) + 1.4(55 + -29) + 1.1(-29+55))$

$= 40.75$ meters

(c) Use graphing calc to graph
$y_1 = 45\sqrt{t}\cos(0.063t^2)$
$y_2 = 60$
look for where $y_1 \geq y_2$
make sure window is set for given interval of $0 \leq t \leq 4$ and a ymin & max that will allow you to find intersection between y_1 & y_2.

Vel of particle Q is at least 60 m/hr in the interval $1.866 \le t \le 3.519$.

[total distance travelled $= \int |v(t)| \, dt$]

distance travelled by Q in this interval is

$$\int_{1.866}^{3.519} \left| 45\sqrt{t} \cos(0.063t^2) \right| dt$$

$= 106.109$ meters

(d) $\underbrace{\int_0^{2.8} v_p(t)\,dt}_{\text{[From part b, we approx this to be 40.75 m]}}$ = displacement of P from $t=0$ to $t=2.8$ ≈ 40.75

When $t=0$, P started at $x=0$ and at at $t=2.8$, P is at $x=40.75$

$\int_0^{2.8} v_q(t)\,dt = 135.938\text{ m}$ displacement of Q from $t=0$ to $t=2.8$

When $t=0$, Q started at $x=-90$ and at $t=2.8$, Q is at $x=45.938$
[$-90+135.938$]

so P & Q are approx 5.188 m apart when $t=2.8$.

[subtract $45.938-40.75$]

③ (a) $\underbrace{\int_{-6}^{-2} f(x)\,dx}_{\text{[what we have to find]}} + \underbrace{\int_{-2}^{5} f(x)\,dx}_{\text{[use area]}} = \int_{-6}^{5} f(x)\,dx = 7$

use area from $x=-2$ to $x=-1$ → $\frac{1}{2}$

use opp of area from $x=-1$ to $x=\frac{1}{2}$ → $-\frac{3}{4}$

use area from $x=\frac{1}{2}$ to $x=2$ → $\frac{9}{4}$

use area from $x=2$ to $x=5$ → $\left(9-\frac{(3)^2\pi}{4}\right)$

$\frac{1}{2} + -\frac{3}{4} + \frac{9}{4} + \left(9 - \frac{(3)^2\pi}{4}\right)$

$$\int_{-6}^{-2} f(x)\,dx = 7 - \left(\tfrac{1}{2} - \tfrac{3}{4} + \tfrac{9}{4} + \left(9 - \tfrac{9\pi}{4}\right)\right)$$

$$7 - \left(11 - \tfrac{9\pi}{4}\right) = -4 + \tfrac{9\pi}{4}$$

(b) $\int_3^5 (2f'(x) + 4)\,dx = 2\int_3^5 f'(x)\,dx + 4\int_3^5 dx$

$= 2(f(5) - f(3)) + 4x\Big]_3^5$

$= 2(0 - (3 - \sqrt{5})) + 4(5-3$

(c) Using the Candidate Test.

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
5	$11 - \frac{9\pi}{4}$

$g'(x) = \frac{d}{dx}\int_{-2}^{x} f(t)\,dt$

$= f(x)$

The abs max of g in $-2 \le x \le 5$ is $11 - \frac{9\pi}{4}$.

(d) $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$

$$= \frac{10 - 3(2)}{1 - \frac{\pi}{4}}$$

$$= \frac{4}{1 - \frac{\pi}{4}}$$

can leave answer like this to save time

(4) (a) [Need to find $\frac{dV}{dt}$ when $h = 4$]

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi \left(r^2 \frac{dh}{dt} + 2r \frac{dr}{dt} h \right)$$

When $h = 4$:

$$\frac{dV}{dt} = \pi \left(1^2 \cdot \frac{-1}{10}\sqrt{4} + 2(1)(0)(4) \right)$$

$$= -\frac{2\pi}{10} \text{ or } \frac{-\pi}{5}$$

The vol of the water is decreasing at a rate of $\frac{2\pi}{10}$ ft^3/s when $h = 4$ ft.

(b) [Need to look at deriv of $\frac{dh}{dt}$]

$$\frac{d^2h}{dt^2} = \frac{-1}{10} \cdot \frac{1}{2} h^{-1/2} \frac{dh}{dt}$$

$$\left. \frac{d^2h}{dt^2} \right|_{h=3} = \frac{-1}{10} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{3}} \cdot \left. \frac{dh}{dt} \right|_{h=3}$$

$$\left. \frac{dh}{dt} \right|_{h=3} \downarrow \; \frac{-1}{10}\sqrt{3}$$

Since $\left. \frac{d^2h}{dt^2} \right|_{h=3}$ is $\oplus$ then the rate of change of h is increasing when $h = 3$ ft.

(c) $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$

$$\frac{dh}{-\frac{1}{10}\sqrt{h}} = dt$$

$$\int -10h^{-1/2}\,dh = \int dt$$

$$\frac{-10 \cdot h^{1/2}}{\frac{1}{2}} = t + c$$

$$-20\sqrt{h} = t + c$$

When $t = 0$, $h = 5$:

$$-20\sqrt{5} = 0 + c$$

$$-20\sqrt{5} = c$$

$$-20\sqrt{h} = t - 20\sqrt{5}$$

$$\sqrt{h} = -\frac{1}{20}t + \sqrt{5}$$

$$h = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$$

⑤ (a) [Set up vertical strip because we have the functions in terms of x.]

$$\text{Area} = \int_0^2 (h(x) - g(x))\,dx$$

$$= \int_0^2 \left(\underbrace{6 - 2(x-1)^2}_{} - (-2 + 3\cos(\tfrac{\pi}{2}x))\right)dx$$

$6 - 2(x^2 - 2x + 1)$
$6 - 2x^2 + 4x - 2$
$4 - 2x^2 + 4x$

[Clean this up before integrating]

$$= \int_0^2 \left(4 - 2x^2 + 4x + 2 - 3\cos(\tfrac{\pi}{2}x)\right)dx$$

$$= \int_0^2 \left(6 - 2x^2 + 4x - 3\cos(\tfrac{\pi}{2}x)\right)dx$$

[Can use u-sub here $u = \frac{\pi}{2}x$ so $\frac{du}{dx} = \frac{\pi}{2}$]

$$= \left(6x - \frac{2x^3}{3} + 2x^2 - 3 \cdot \frac{2}{\pi}\sin(\tfrac{\pi}{2}x)\right)\Big]_0^2$$

$$= \left(12 - \frac{16}{3} + 8 - \frac{6}{\pi}\sin\pi\right) - \left(0 - \frac{6}{\pi}\sin 0\right)$$

$$= \left(12 - \frac{16}{3} + 8 - 0\right) - 0$$

$$= 12 - \frac{16}{3} + 8$$

$$= \frac{44}{3}$$

[Can leave answer at this step - do not need to simplify to this]

(b) [to get volume → integrate area of cross section]

$$V = \int_0^2 \left(\frac{1}{x+3}\right) dx$$

$$= \ln|x+3| \Big]_0^2$$

$$= \ln 5 - \ln 3 \quad \text{or} \quad \ln \frac{5}{3}$$

(c) [Use same vertical strip as for part (a) and use donut (washer) method]

$$V = \pi \int_0^2 \left((g(x)-6)^2 - (h(x)-6)^2 \right) dx$$

[Leave this as your answer. Question said "set up, but do not integrate."]

⑥ (a)

$h'(2) = \frac{2}{3} \leftarrow$ [slope of tan line to h when $x=2$]

(b) $a'(x) = 3x^3 \cdot h'(x) + 9x^2 h(x)$

$a'(2) = 3(2)^3 \cdot h'(2) + 9(2)^2 h(2)$

$= 3 \cdot 8 \cdot \frac{2}{3} + 9 \cdot 4 \cdot 4$

Using L'Hospital's Rule:

(c) $\underbrace{\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3}}_{4} = \lim_{x\to 2} \frac{2x}{-3(f(x))^2 \cdot f'(x)}$

so $\rightarrow 4 = \frac{2(2)}{-3(f(2))^2 \cdot f'(2)}$

to get $f(2)$

Because h is diff then h is cont

so $h(2) = \lim_{x\to 2} h(x)$

Since L'Hosp. Rule applies

and $\lim_{x\to 2} x^2-4 = 0$ then $\lim_{x\to 2} 1-(f(x))^3 = 0$

so $\lim_{x\to 2} f(x) = 1$

Since f is diff + therefore cont. $f(2) = 1$

so $4 = \dfrac{4}{-3(1)^2 \cdot f'(2)}$

$$4 = \frac{4}{-3f'(2)}$$

$$\frac{1}{4} = \frac{-3f'(2)}{4}$$

$$1 = -3f'(2)$$

$$-\frac{1}{3} = f'(2)$$

(d) For k is be cont. at $x = 2$ then

$$k(2) = \lim_{x\to 2} k(x)$$

$g(2) = 4$

$h(2) = 4$

so if

$g(x) \le k(x) \le h(x)$

for $1 < x < 3$

then

$k(2) = 4$

Since g & h are diff then g & h are cont.

so $\lim\limits_{x\to 2} g(x) = g(2) = 4$

and

$\lim\limits_{x\to 2} h(x) = h(2) = 4$

This along with

$g(x) \le k(x) \le h(x)$

for $1 < x < 3$

means $\lim\limits_{x\to 2} k(x) = 4$

$\therefore$ k is cont at $x=2$.