Name:__________________________ Date:___________

PCH: Trigonometric Identities and Proofs Ms. Loughran

The Pythagorean Identities:

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

Double Angle Formulas:

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2x = \begin{matrix} \cos^2 x - \sin^2 x \\ 1 - 2\sin^2 x \\ 2\cos^2 x - 1 \end{matrix}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

You are familiar with the following reciprocal identities:

$$\sec\theta = \frac{1}{\cos\theta}, \cos\theta \neq 0 \qquad \csc\theta = \frac{1}{\sin\theta}, \sin\theta \neq 0 \qquad \cot\theta = \frac{1}{\tan\theta}, \tan\theta \neq 0$$

And the quotient identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cos\theta \neq 0 \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}, \sin\theta \neq 0$$

An identity is an equation that is true for all permissible replacements of the variable.

Proving an identity:
To prove that a trigonometric statement is an identity, note:

1. **The object is to show that the two sides of the statement are equivalent.**
   - ⇒ You may work on only one side and show that it is equivalent to the other.
     - ➢ Work on the more complicated side.
   - ⇒ You may work on the two sides independently until you arrive at equivalent expressions.
     - ➢ You may not perform operations involving the two sides simultaneously. You are not solving an equation. That is, never cross the equal sign for any purpose. As a reminder, use a line between sides.

2. **Use the basic identities to transform one or both sides of the proposed identity.**
   - ⇒ A general starting point is to rewrite expressions in terms of sine and cosine, but be alert to situations when a Pythagorean substitution is appropriate.

3. **After replacements have been made, do the algebra suggested by the form of the expression.**
   - ⇒ If there is a complex fraction, simplify it.
   - ⇒ If there are two fractions, combine them.
   - ⇒ Look for possibilities of factoring.

Classwork

1. Simplify the expression: $\cos t + \tan t \sin t$

2. Simplify the expression: $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta}$

3. Verify the identity: $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$

4. Verify the identity: $2\tan x \sec x = \dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x}$

5. Verify the identity: $\dfrac{\cos u}{1-\sin u} = \sec u + \tan u$

6. Verify the identity: $\frac{1+\cos\theta}{\cos\theta}=\frac{\tan^2\theta}{\sec\theta-1}$

7. Verify the identity: $(\sin x+\cos x)^2=1+\sin 2x$

7.1 Exercises

1–10 ■ Write the trigonometric expression in terms of sine and cosine, and then simplify.

1. $\cos t \tan t$

2. $\cos t \csc t$

3. $\sin \theta \sec \theta$

4. $\tan \theta \csc \theta$

5. $\tan^2 x - \sec^2 x$

6. $\dfrac{\sec x}{\csc x}$

7. $\sin u + \cot u \cos u$

8. $\cos^2\theta\,(1 + \tan^2\theta)$

9. $\dfrac{\sec \theta - \cos \theta}{\sin \theta}$

10. $\dfrac{\cot \theta}{\csc \theta - \sin \theta}$

11–24 ■ Simplify the trigonometric expression.

11. $\dfrac{\sin x \sec x}{\tan x}$

12. $\cos^3 x + \sin^2 x \cos x$

13. $\dfrac{1 + \cos y}{1 + \sec y}$

14. $\dfrac{\tan x}{\sec(-x)}$

15. $\dfrac{\sec^2 x - 1}{\sec^2 x}$

16. $\dfrac{\sec x - \cos x}{\tan x}$

17. $\dfrac{1 + \csc x}{\cos x + \cot x}$

18. $\dfrac{\sin x}{\csc x} + \dfrac{\cos x}{\sec x}$

19. $\dfrac{1 + \sin u}{\cos u} + \dfrac{\cos u}{1 + \sin u}$

20. $\tan x \cos x \csc x$

21. $\dfrac{2 + \tan^2 x}{\sec^2 x} - 1$

22. $\dfrac{1 + \cot A}{\csc A}$

23. $\tan \theta + \cos(-\theta) + \tan(-\theta)$

24. $\dfrac{\cos x}{\sec x + \tan x}$

25–88 ■ Verify the identity.

25. $\dfrac{\sin \theta}{\tan \theta} = \cos \theta$

26. $\dfrac{\tan x}{\sec x} = \sin x$

27. $\dfrac{\cos u \sec u}{\tan u} = \cot u$

28. $\dfrac{\cot x \sec x}{\csc x} = 1$

29. $\dfrac{\tan y}{\csc y} = \sec y - \cos y$

30. $\dfrac{\cos v}{\sec v \sin v} = \csc v - \sin v$

31. $\sin B + \cos B \cot B = \csc B$

32. $\cos(-x) - \sin(-x) = \cos x + \sin x$

33. $\cot(-\alpha)\cos(-\alpha) + \sin(-\alpha) = -\csc \alpha$

34. $\csc x\,[\csc x + \sin(-x)] = \cot^2 x$

35. $\tan \theta + \cot \theta = \sec \theta \csc \theta$

36. $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$

37. $(1 - \cos \beta)(1 + \cos \beta) = \dfrac{1}{\csc^2 \beta}$

38. $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$

39. $\dfrac{(\sin x + \cos x)^2}{\sin^2 x - \cos^2 x} = \dfrac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}$

40. $(\sin x + \cos x)^4 = (1 + 2 \sin x \cos x)^2$

41. $\dfrac{\sec t - \cos t}{\sec t} = \sin^2 t$

42. $\dfrac{1 - \sin x}{1 + \sin x} = (\sec x - \tan x)^2$

43. $\dfrac{1}{1 - \sin^2 y} = 1 + \tan^2 y$

44. $\csc x - \sin x = \cos x \cot x$

45. $(\cot x - \csc x)(\cos x + 1) = -\sin x$

46. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

47. $(1 - \cos^2 x)(1 + \cot^2 x) = 1$

48. $\cos^2 x - \sin^2 x = 2 \cos^2 x - 1$

49. $2 \cos^2 x - 1 = 1 - 2 \sin^2 x$

50. $(\tan y + \cot y) \sin y \cos y = 1$

51. $\dfrac{1 - \cos \alpha}{\sin \alpha} = \dfrac{\sin \alpha}{1 + \cos \alpha}$

52. $\sin^2\alpha + \cos^2\alpha + \tan^2\alpha = \sec^2\alpha$

53. $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$

54. $\cot^2\theta \cos^2\theta = \cot^2\theta - \cos^2\theta$

55. $\dfrac{\sin x - 1}{\sin x + 1} = \dfrac{-\cos^2 x}{(\sin x + 1)^2}$

56. $\dfrac{\sin w}{\sin w + \cos w} = \dfrac{\tan w}{1 + \tan w}$

57. $\dfrac{(\sin t + \cos t)^2}{\sin t \cos t} = 2 + \sec t \csc t$

58. $\sec t \csc t\,(\tan t + \cot t) = \sec^2 t + \csc^2 t$

59. $\dfrac{1 + \tan^2 u}{1 - \tan^2 u} = \dfrac{1}{\cos^2 u - \sin^2 u}$

60. $\dfrac{1 + \sec^2 x}{1 + \tan^2 x} = 1 + \cos^2 x$

61. $\dfrac{\sec x}{\sec x - \tan x} = \sec x\,(\sec x + \tan x)$

62. $\dfrac{\sec x + \csc x}{\tan x + \cot x} = \sin x + \cos x$

63. $\sec v - \tan v = \dfrac{1}{\sec v + \tan v}$

64. $\dfrac{\sin A}{1 - \cos A} - \cot A = \csc A$

65. $\dfrac{\sin x + \cos x}{\sec x + \csc x} = \sin x \cos x$

66. $\dfrac{1 - \cos x}{\sin x} + \dfrac{\sin x}{1 - \cos x} = 2 \csc x$

67. $\dfrac{\csc x - \cot x}{\sec x - 1} = \cot x$

68. $\dfrac{\csc^2 x - \cot^2 x}{\sec^2 x} = \cos^2 x$

69. $\tan^2 u - \sin^2 u = \tan^2 u \sin^2 u$

70. $\dfrac{\tan v \sin v}{\tan v + \sin v} = \dfrac{\tan v - \sin v}{\tan v \sin v}$

71. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

72. $\dfrac{\cos\theta}{1 - \sin\theta} = \sec\theta + \tan\theta$

73. $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{\sin\theta - \csc\theta}{\cos\theta - \cot\theta}$

74. $\dfrac{1 + \tan x}{1 - \tan x} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

75. $\dfrac{\cos^2 t + \tan^2 t - 1}{\sin^2 t} = \tan^2 t$

76. $\dfrac{1}{1 - \sin x} - \dfrac{1}{1 + \sin x} = 2 \sec x \tan x$

77. $\dfrac{1}{\sec x + \tan x} + \dfrac{1}{\sec x - \tan x} = 2 \sec x$

78. $\dfrac{1 + \sin x}{1 - \sin x} - \dfrac{1 - \sin x}{1 + \sin x} = 4 \tan x \sec x$

79. $(\tan x + \cot x)^2 = \sec^2 x + \csc^2 x$

80. $\tan^2 x - \cot^2 x = \sec^2 x - \csc^2 x$

81. $\dfrac{\sec u - 1}{\sec u + 1} = \dfrac{1 - \cos u}{1 + \cos u}$

82. $\dfrac{\cot x + 1}{\cot x - 1} = \dfrac{1 + \tan x}{1 - \tan x}$

83. $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$

84. $\dfrac{\tan v - \cot v}{\tan^2 v - \cot^2 v} = \sin v \cos v$

85. $\dfrac{1 + \sin x}{1 - \sin x} = (\tan x + \sec x)^2$

86. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$

87. $(\tan x + \cot x)^4 = \csc^4 x \sec^4 x$

88. $(\sin\alpha - \tan\alpha)(\cos\alpha - \cot\alpha) = (\cos\alpha - 1)(\sin\alpha - 1)$

89–94 ■ Make the indicated trigonometric substitution in the given algebraic expression and simplify (see Example 7). Assume $0 \le \theta < \pi/2$.

89. $\dfrac{x}{\sqrt{1 - x^2}}, \quad x = \sin\theta$

90. $\sqrt{1 + x^2}, \quad x = \tan\theta$

91. $\sqrt{x^2 - 1}, \quad x = \sec\theta$

92. $\dfrac{1}{x^2\sqrt{4 + x^2}}, \quad x = 2\tan\theta$

93. $\sqrt{9 - x^2}, \quad x = 3\sin\theta$

94. $\dfrac{\sqrt{x^2 - 25}}{x}, \quad x = 5\sec\theta$

95–98 ■ Graph f and g in the same viewing rectangle. Do the graphs suggest that the equation $f(x) = g(x)$ is an identity? Prove your answer.

95. $f(x) = \cos^2 x - \sin^2 x, \quad g(x) = 1 - 2\sin^2 x$

96. $f(x) = \tan x\,(1 + \sin x), \quad g(x) = \dfrac{\sin x \cos x}{1 + \sin x}$

97. $f(x) = (\sin x + \cos x)^2, \quad g(x) = 1$

98. $f(x) = \cos^4 x - \sin^4 x, \quad g(x) = 2\cos^2 x - 1$

99. Show that the equation is not an identity.

(a) $\sin 2x = 2 \sin x$

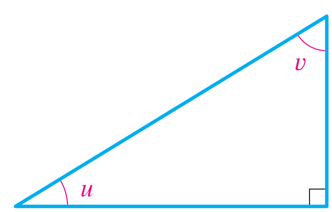
(b) $\sin(x + y) = \sin x + \sin y$

(c) $\sec^2 x + \csc^2 x = 1$

(d) $\dfrac{1}{\sin x + \cos x} = \csc x + \sec x$

Discovery • Discussion

100. Cofunction Identities In the right triangle shown, explain why $v = (\pi/2) - u$. Explain how you can obtain all six cofunction identities from this triangle, for $0 < u < \pi/2$.



101. Graphs and Identities Suppose you graph two functions, f and g, on a graphing device, and their graphs appear identical in the viewing rectangle. Does this prove that the equation $f(x) = g(x)$ is an identity? Explain.

102. Making Up Your Own Identity If you start with a trigonometric expression and rewrite it or simplify it, then setting the original expression equal to the rewritten expression yields a trigonometric identity. For instance, from Example 1 we get the identity

$$\cos t + \tan t \sin t = \sec t$$

Use this technique to make up your own identity, then give it to a classmate to verify.